Published by Institute of Physics Publishing for SISSA

Received: *July 17, 2008*
Accepted: *August 11, 2008*
Published: *September 1, 2008*

$\mathcal{N} = 5, 6$ superconformal Chern-Simons theories and M2-branes on orbifolds

Kazuo Hosomichi,[a] Ki-Myeong Lee,[a] Sangmin Lee,[b] Sungjay Lee[a] and Jaemo Park[cde]

[a] *Korea Institute for Advanced Study, Seoul 130-012, Korea*
[b] *Department of Physics & Astronomy, Seoul National University, Seoul 151-747, Korea*
[c] *Department of Physics, Postech, Pohang 790-784, Korea*
[d] *Postech Center for Theoretical Physics (PCTP), Postech, Pohang 790-784, Korea*
[e] *Department of Physics, Stanford University, Stanford, CA 94305-4060, U.S.A.*
E-mail: hosomiti@kias.re.kr, klee@kias.re.kr, sangmin@snu.ac.kr, sjlee@kias.re.kr, jaemo@postech.ac.kr

Abstract: We explore further our recent generalization of the $\mathcal{N} = 4$ superconformal Chern-Simons theories of Gaiotto and Witten. We find and construct explicitly theories of enhanced $\mathcal{N} = 5$ or 6 supersymmetry, especially $\mathcal{N} = 5$, $\mathrm{Sp}(2M) \times O(N)$ and $\mathcal{N} = 6$, $\mathrm{Sp}(2M) \times O(2)$ theories. The $\mathrm{U}(M) \times \mathrm{U}(N)$ theory coincides with the $\mathcal{N} = 6$ theory of Aharony, Bergman, Jafferis and Maldacena (ABJM). We argue that the $\mathcal{N} = 5$ theory with $\mathrm{Sp}(2N) \times O(2N)$ gauge group can be understood as an orientifolding of the ABJM model with $\mathrm{U}(2N) \times \mathrm{U}(2N)$ gauge group. We briefly discuss the Type IIB brane construction of the $\mathcal{N} = 5$ theory and the geometry of the M-theory orbifold.

Keywords: Field Theories in Lower Dimensions, AdS-CFT Correspondence, M-Theory, Brane Dynamics in Gauge Theories.

JHEP09(2008)002

Contents

JHEP09(2008)002

1. Introduction and concluding remarks

Superconformal Chern-Simons theories have become a subject of intensive research recently. Schwarz [1] suggested that Chern-Simons theories without Yang-Mills kinetic term may be used to describe the $\mathcal{N} = 8$ superconformal M2-brane world-volume theory. The idea was crystalized by Bagger and Lambert [2–4] and Gustavsson [5, 6] (BLG) who proposed an $\mathcal{N} = 8$ Chern-Simons theory based on 3-algebra, and gave an explicit example with $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge group. The $\mathrm{SO}(4)$ BLG theory can be rewritten as an ordinary gauge theory [7, 8]. It is conjectured to be a specific theory of some M2 brane configuration [9, 10].

More recently, Aharony, Bergman, Jafferis and Maldacena (ABJM) [11] have found $\mathcal{N} = 6$ superconformal Chern-Simons theories of $\mathrm{U}(M) \times \mathrm{U}(N)$ gauge group, and have argued that the theories with $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge group and Chern-Simons level k is the holographic dual of the M-theory geometry background arising from N M2 branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. Especially for $k = 1, 2$ cases, it has been argued that the supersymmetry is enhanced to $\mathcal{N} = 8$.

On the other hand, with somewhat different motivation, Gaiotto and Witten [12] have constructed a class of $\mathcal{N} = 4$ superconformal Chern-Simons theories coupled to hyper-multiplets. The gauge group and matter content are severally restricted and determined with the help of general classification of Lie super-algebra. In particular, the theory can come with $\mathrm{U}(M) \times \mathrm{U}(N)$ or $\mathrm{Sp}(2M) \times O(N)$ gauge group and a single bi-fundamental hyper-multiplet. In a subsequent work [13], we have constructed more general $\mathcal{N} = 4$ linear quiver-type theories where bi-fundamental hyper and twisted multiplets alternate between connected nodes. The special case of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge group with both hyper and twisted hyper-multiplets becomes identical to the BLG theory with enhanced $\mathcal{N} = 8$ supersymmetry. It was also suggested in [13] that the embedding of the Chern-Simons theories into IIB string theory [12] can be T-dualized to make contact with M2-branes on orbifolds.

In this work, we show by an explicit construction that the $\mathcal{N} = 4$ theories with two gauge groups and both hyper and twisted hyper-multiplets in the same gauge representation always have an enhanced $\mathcal{N} = 5$ or more supersymmetry. Especially those with $\mathrm{U}(M) \times \mathrm{U}(N)$ gauge group coincides with the $\mathcal{N} = 6$ ABJM theory. We also find new $\mathcal{N} = 5$ theories of $\mathrm{Sp}(2M) \times O(N)$ $(N > 2)$ and $\mathcal{N} = 6$ theories of $\mathrm{Sp}(2M) \times O(2)$. We argue that the $\mathrm{Sp}(2N) \times O(2N)$ theory can be obtained by a simple orientifolding of $\mathcal{N} = 6$ $\mathrm{U}(2N) \times \mathrm{U}(2N)$ ABJM theory and can be regarded as the holographic dual of the M-theory geometry for M2 branes exploring the orbifold $\mathbb{C}^4/D_k$ where $2k - 2$ is the level of the Chern-Simons coefficient.

The $\mathcal{N} = 3$ theories can come with arbitrary gauge group and matter hyper-multiplets, and are not subject to any quantum correction to the Chern-Simons level. The superconformal theory of $\mathcal{N} = 2, 3$ theories has been studied extensively by Gaiotto and Yin [14]. ABJM have shown that the $\mathcal{N} = 3$ theory with $\mathrm{U}(M) \times \mathrm{U}(N)$ gauge group and a 'pair' of bi-fundamental matter field have an enhanced $\mathcal{N} = 6$ supersymmetry, by arguing that the theory has a $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry which does not commute with the $\mathrm{SU}(2)$ R-symmetry. The ABJM theory falls into our category of theories with enhanced supersymmetry.

We begin with a brief summary of $\mathcal{N} = 4$ theories with both types of hyper-multiplets in section 2, and then show by an explicit construction that whenever the hyper-multiplets belong to the same gauge representation, there is an enhancement of the supersymmetry to at least $\mathcal{N} = 5$. Then we work out the Lagrangian for the theories of $\mathrm{Sp}(2M) \times O(N)$ gauge group in detail.

In section 3, we study a further enhancement of $\mathcal{N} = 5$ to $\mathcal{N} = 6$ supersymmetry, which occurs whenever the matter representation can be decomposed a complex representation (R) and its complex conjugate representation $(\bar{R})$, or the matter representation is purely real. The $\mathrm{U}(M) \times \mathrm{U}(N)$ gauge theory becomes the ABJM model, and the $\mathrm{Sp}(2M) \times O(2)$ gauge theory provides a new example of $\mathcal{N} = 6$ superconformal field theories.

We could pursue our analysis further and construct general $\mathcal{N} = 7, 8$ theories as well. From the $\mathcal{N} = 6$ point of view, enhancement of supersymmetry to $\mathcal{N} \geq 7$ requires that the representation R be real; the $\mathrm{SO}(4)$ BLG theory is such an example. It should be a simple matter to find or rule out any possibility other than the BLG theory by examining the classification of Lie super-algebra, but we will not pursue it in this paper.

Given the IIB brane configuration of the ABJM model, it is easy to relate our new $\mathcal{N} = 5$ OSp$(M|N)$ theories to an orientifold of the ABJM model. We elaborate on this point in section 4. Taking the T-duality to M-theory as in the ABJM model, we obtain a D_k orbifold $\mathbb{C}^4$. Orbifolds of $\mathbb{C}^4$ preserving $\mathcal{N} = 5$ supersymmetry were discussed earlier in [15] and very recently in [16].

In appendices, we provide the detailed computations for the $\mathcal{N} = 5, 6$ cases and express the mass deformation of the $\mathcal{N} = 4$ Lagrangian [13] in $\mathcal{N} = 5, 6$ language. It shows that all of the $\mathcal{N} = 5, 6$ supersymmetries are preserved, while the SO(5) or SO(6) R-symmetries are partially broken to SO(4) or SO(4) $\times$ U(1), respectively. This is somewhat expected from the mass deformation of the BLG theory [17, 18].

We close this introduction with some directions for further study. The 3-algebra structure played a crucial role in making the BLG model compatible with $\mathcal{N} = 8$ supersymmetry, while the ABJM model at $k = 1$ is argued to have $\mathcal{N} = 8$ without using the 3-algebra structure. It would be desirable to understand the relation between the two approaches. A recent paper [19] has taken a step in this direction (see also [20 − 26]). More work is required to establish the AdS/CFT correspondence of the Chern-Simons theories. The relevant topics include extension to $\mathcal{N} = 4$ orbifolds [27, 28], superconformal indices [29], Penrose limit [30], integrability [31], non-supersymmetric generalization [32], and partition function [16]. Finally, it remains to derive, from first principles, much richer family of $\mathcal{N} \geq 2$ superconformal theories with known M-theory geometry [27, 33 − 37].

2. $\mathcal{N} = 5$ superconformal theories

2.1 Review of $\mathcal{N} = 4$

We review the construction of general $\mathcal{N} = 4$ superconformal Chern-Simons-matter theories [12, 13]. We start with an Sp$(2n)$ group and let A, B indices run over a $2n$-dimensional representation. We denote the anti-symmetric invariant tensor of Sp$(2n)$ by ω_{AB} and choose all the generators $t^A{}_B$ to be anti-Hermitian $(2n \times 2n)$ matrices such that $t_{AB} = \omega_{AC} t^C{}_B$ are symmetric matrices. We consider a Chern-Simons gauge theory whose gauge group is a subgroup of Sp$(2n)$ and denote anti-Hermitian generators of the gauge group as $(t^m)^A{}_B$ which satisfy

$$[t^m, t^n] = f^{mn}{}_p t^p.$$

The gauge field is denoted by $(A_m)_\mu$, and the adjoint indices are raised or lowered by an invariant quadratic form k^{mn} or its inverse k_{mn} of the gauge group.

We couple the gauge theory with hyper and twisted hyper-multiplet matter fields $(q^A_\alpha, \psi^A_{\dot\alpha}; \tilde q^A_{\dot\alpha}, \tilde\psi^A_\alpha)$ satisfying the reality conditions

$$\bar q^\alpha_A = (q^A_\alpha)^\dagger = \epsilon^{\alpha\beta}\omega_{AB} q^B_\beta, \qquad \bar\psi^A_{\dot\alpha} = (\psi^A_{\dot\alpha})^\dagger = \epsilon^{\dot\alpha\dot\beta}\omega_{AB}\psi^B_{\dot\beta}, \tag{2.1}$$

and similar conditions for $\tilde q^A_{\dot\alpha}$ and $\tilde\psi^A_\alpha$. We use $(\alpha, \beta; \dot\alpha, \dot\beta)$ doublet indices for the SU$(2)_L \times$ SU$(2)_R$ R-symmetry group. We also have the inverse tensors, $\omega^{AB}, \epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}$ such that, say, $\omega_{AC}\omega^{CB} = \delta_A{}^B$, and $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$.

Both types of hyper-multiplets share the same gauge symmetry, so the structure constants $f^{mn}{}_p$ and the quadratic form k^{mn} are identical. But, they can take different representations in general $\mathcal{N}=4$ theories. For $\mathcal{N}>4$ supersymmetry, however, the two types of hyper-multiplets should be combined together into a bigger multiplet, so they have to take the same representation.

In the construction of ref. [12, 13], $\mathcal{N}=1$ super-field formulation was used and conditions on the super-potentials for enhancement to $\mathcal{N}=4$ was examined. It was found that there is essentially one constraint equation (called "fundamental identity" in [12]),

$$k_{mn} t^m_{A(B} t^n_{CD)} \;=\; 0, \tag{2.2}$$

where the expression is summed over the cyclic permutation of indices B, C, D. When this condition is satisfied, all $\mathcal{N}=1$ super-potentials are uniquely determined, and we end up with an $\mathcal{N}=4$ theory.

Following [12], we introduce the "moment map" and "current" operators,

$$\begin{aligned}
&\mu^m_{\alpha\beta} \equiv t^m_{AB} q^A_\alpha q^B_\beta, && \mu^{mn}_{\alpha\beta} \equiv (t^m t^n)_{AB} q^A_\alpha q^B_\beta, && \jmath^m_{\alpha\dot\alpha} \equiv q^A_\alpha t^m_{AB} \psi^B_{\dot\alpha}, \\
&\tilde\mu^m_{\dot\alpha\dot\beta} \equiv \tilde t^m_{AB} \tilde q^A_{\dot\alpha} \tilde q^B_{\dot\beta}, && \tilde\mu^{mn}_{\dot\alpha\dot\beta} \equiv (\tilde t^m \tilde t^n)_{AB} \tilde q^A_{\dot\alpha} \tilde q^B_{\dot\beta}, && \tilde\jmath^m_{\dot\alpha\alpha} \equiv \tilde q^A_{\dot\alpha} \tilde t^m_{AB} \tilde\psi^B_\alpha .
\end{aligned} \tag{2.3}$$

Using these notations, we can write down the Lagrangian of general $\mathcal{N}=4$ superconformal Chern-Simons-matter theories in a fully covariant form [13],

$$\begin{aligned}
\mathcal{L} = & \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn} A^m_\mu \partial_\nu A^n_\lambda + \frac{1}{3} f_{mnp} A^m_\mu A^n_\nu A^p_\lambda\right) \\
&+\frac{1}{2}\omega_{AB}\left(-\epsilon^{\alpha\beta} D q^A_\alpha D q^B_\beta + i\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha} \not{D} \psi^B_{\dot\beta}\right) + \frac{1}{2}\omega_{AB}\left(-\epsilon^{\dot\alpha\dot\beta} D\tilde q^A_{\dot\alpha} D\tilde q^B_{\dot\beta} + i\epsilon^{\alpha\beta}\tilde\psi^A_\alpha \not{D}\tilde\psi^B_\beta\right) \\
&-i\pi k_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}\jmath^m_{\alpha\dot\gamma}\jmath^n_{\beta\dot\delta} - i\pi k_{mn}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\gamma\delta}\tilde\jmath^m_{\dot\alpha\gamma}\tilde\jmath^n_{\dot\beta\delta} + 4\pi i k_{mn}\epsilon^{\alpha\gamma}\epsilon^{\dot\beta\dot\delta}\jmath^m_{\alpha\dot\beta}\tilde\jmath^n_{\dot\delta\gamma} \\
&+i\pi k_{mn}\left(\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}\tilde\mu^m_{\dot\alpha\dot\beta}\psi^A_{\dot\gamma} t^n_{AB}\psi^B_{\dot\delta} + \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mu^m_{\alpha\beta}\tilde\psi^A_\gamma \tilde t^n_{AB}\tilde\psi^B_\delta\right) \\
&-\frac{\pi^2}{6} f_{mnp}(\mu^m)^\alpha{}_\beta(\mu^n)^\beta{}_\gamma(\mu^p)^\gamma{}_\alpha - \frac{\pi^2}{6} f_{mnp}(\tilde\mu^m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu^n)^{\dot\beta}{}_{\dot\gamma}(\tilde\mu^p)^{\dot\gamma}{}_{\dot\alpha} \\
&+\pi^2(\tilde\mu^{mn})^{\dot\gamma}{}_{\dot\gamma}(\mu_m)^\alpha{}_\beta(\mu_n)^\beta{}_\alpha + \pi^2(\mu^{mn})^\gamma{}_\gamma(\tilde\mu_m)^{\dot\alpha}{}_{\dot\beta}(\tilde\mu_n)^{\dot\beta}{}_{\dot\alpha} .
\end{aligned} \tag{2.4}$$

The supersymmetry transformation law reads,

$$\begin{aligned}
&\delta q^A_\alpha = +i\eta_\alpha{}^{\dot\alpha}\psi^A_{\dot\alpha}, \quad \delta\tilde q^A_{\dot\alpha} = -i\eta^\alpha{}_{\dot\alpha}\tilde\psi^A_\alpha, \quad \delta A^m_\mu = 2\pi i\eta^{\alpha\dot\alpha}\gamma_\mu(\jmath^m_{\alpha\dot\alpha} - \tilde\jmath^m_{\dot\alpha\alpha}), \\
&\delta\psi^A_{\dot\alpha} = +\left[\not{D} q^A_\alpha + \frac{2\pi}{3}(t_m)^A{}_B q^B_\beta(\mu^m)^\beta{}_\alpha\right]\eta^\alpha{}_{\dot\alpha} - 2\pi(t_m)^A{}_B q^B_\beta(\tilde\mu^m)^{\dot\beta}{}_{\dot\alpha}\eta^\beta{}_{\dot\beta}, \\
&\delta\tilde\psi^A_\alpha = -\left[\not{D}\tilde q^A_{\dot\alpha} + \frac{2\pi}{3}(\tilde t_m)^A{}_B\tilde q^B_{\dot\beta}(\tilde\mu^m)^{\dot\beta}{}_{\dot\alpha}\right]\eta_\alpha{}^{\dot\alpha} + 2\pi(\tilde t_m)^A{}_B\tilde q^B_{\dot\beta}(\mu^m)^\beta{}_\alpha\eta_\beta{}^{\dot\beta}.
\end{aligned} \tag{2.5}$$

The spinor parameter $\eta_{\alpha\dot\beta}$ satisfies the reality condition

$$(\eta_\alpha{}^{\dot\beta})^* = \eta^\alpha{}_{\dot\beta} = \epsilon^{\alpha\beta}\epsilon_{\dot\beta\dot\alpha}\eta_\beta{}^{\dot\alpha} . \tag{2.6}$$

In ref. [12], it was noticed that the fundamental identity can be understood as the Jacobi identity for three fermionic generators of a Lie super-algebra,

$$[M^m, M^n] = f^{mn}{}_p M^p, \qquad [M^m, Q_A] = Q_B (t^m)^B{}_A, \qquad \{Q_A, Q_B\} = t^m_{AB} M_m \,. \tag{2.7}$$

This turns out to be a rather strong constraint on the field content of the theory. Namely, the gauge group and matter should be such that the gauge symmetry algebra can be extended to a Lie super-algebra by adding fermionic generators associated to hyper-multiplets.

The notion of Lie super-algebra characterizing $\mathcal{N}=4$ theories will be useful throughout the rest of the paper, as we investigate the conditions for enhanced supersymmetry ($\mathcal{N}>4$).

2.2 General construction

A necessary condition for supersymmetry enhancement is that the two types of hyper-multiplets in the $\mathcal{N}=4$ theory take the same representation of the gauge group. In this section, we will show that this is also sufficient for enhancement to $\mathcal{N}=5$. In other words, for *any* (extended) $\mathcal{N}=4$ Gaiotto-Witten theory, if the two types of hyper-multiplets are in the same representation of the gauge group so that $t^m_{AB} = \tilde{t}^m_{AB}$, the supersymmetry is *automatically* enhanced to $\mathcal{N}=5$.

The lift from $\mathcal{N}=4$ to $\mathcal{N}=5$ is an exercise of embedding the R-symmetry group $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{Sp}(2)\times\mathrm{Sp}(2)$ into $\mathrm{Sp}(4) = \mathrm{SO}(5)$ in the standard way. We combine the $\mathcal{N}=4$ hyper and twisted hyper-multiplets to form $\mathcal{N}=5$ multiplets,

$$\Phi^A_\alpha = \begin{pmatrix} q^A_\alpha \\ \tilde{q}^A_{\dot\alpha} \end{pmatrix}, \qquad \Psi^A_\alpha = \begin{pmatrix} \tilde{\psi}^A_\alpha \\ \psi^A_{\dot\alpha} \end{pmatrix}. \tag{2.8}$$

The reality conditions can be rewritten in the $\mathcal{N}=5$ covariant way as

$$\bar{\Phi}^\alpha_A = (\Phi^A_\alpha)^\dagger = C^{\alpha\beta}\omega_{AB}\Phi^B_\beta \,, \qquad \bar{\Psi}^A_\alpha = (\Psi^A_\alpha)^\dagger = C^{\alpha\beta}\omega_{AB}\Psi^B_\beta \,, \tag{2.9}$$

where the invariant tensor of $\mathrm{Sp}(4)$,

$$C^{\alpha\beta} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \tag{2.10}$$

can be understood as the charge conjugation matrix for the $\mathrm{SO}(5)$ Clifford algebra in a suitably chosen basis. The "moment map" and the "current" operators also take the $\mathcal{N}=5$ form,

$$\mathcal{M}^m_{\alpha\beta} \equiv t^m_{AB}\Phi^A_\alpha\Phi^B_\beta \,, \qquad \mathcal{M}^{mn}_{\alpha\beta} \equiv (t^m t^n)_{AB}\Phi^A_\alpha\Phi^B_\beta \,, \qquad \mathcal{J}^m_{\alpha\beta} \equiv t^m_{AB}\Phi^A_\alpha\Psi^B_\beta \,. \tag{2.11}$$

After a slightly lengthy algebra (see appendix A), we can uplift the $\mathcal{N}=4$ Lagrangian (2.4) in the $\mathcal{N}=5$ Lagrangian,

$$\begin{aligned}\mathcal{L} = {}& \frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right) \\ & + \frac{1}{2}\omega_{AB}C^{\alpha\beta}\left(-D\Phi^A_\alpha D\Phi^B_\beta + i\Psi^A_\alpha \not{D}\Psi^B_\beta\right) - i\pi k_{mn}C^{\alpha\beta}C^{\gamma\delta}\left(\mathcal{J}^m_{\alpha\gamma}\mathcal{J}^n_{\beta\delta} - 2\mathcal{J}^m_{\alpha\gamma}\mathcal{J}^n_{\delta\beta}\right) \\ & + \frac{2\pi^2}{15}f_{mnp}(\mathcal{M}^m)^\alpha{}_\beta(\mathcal{M}^n)^\beta{}_\gamma(\mathcal{M}^p)^\gamma{}_\alpha + \frac{3\pi^2}{5}(\mathcal{M}^{mn})^\gamma{}_\gamma(\mathcal{M}_m)^\alpha{}_\beta(\mathcal{M}_n)^\beta{}_\alpha \,,\end{aligned} \tag{2.12}$$

JHEP09(2008)002

and the supersymmetry transformation law,

$$\delta\Phi^A_\alpha = i\eta_\alpha{}^\beta\Psi^A_\beta\,,\quad \delta A^m_\mu = 2\pi i\eta^{\alpha\beta}\gamma_\mu\mathcal{J}^m_{\alpha\beta}\,,$$
$$\delta\Psi^A_\alpha = \left[\not{D}\Phi^A_\gamma + \frac{2\pi}{3}(t_m)^A{}_B\Phi^B_\beta(\mathcal{M}^m)^\beta{}_\gamma\right]\eta^\gamma{}_\alpha - \frac{4\pi}{3}(t_m)^A{}_B\Phi^B_\beta(\mathcal{M}^m)^\gamma{}_\alpha\eta^\beta{}_\gamma\,. \tag{2.13}$$

The parameter $\eta_{\alpha\beta}$ satisfies

$$\eta_{\alpha\beta} = -\eta_{\beta\alpha}\,,\qquad (\eta^*)^{\alpha\beta} = -C^{\alpha\gamma}C^{\beta\delta}\eta_{\gamma\delta}\,,\qquad C^{\alpha\beta}\eta_{\alpha\beta} = 0\,. \tag{2.14}$$

2.3 OSp($N|2M$) example

Symplectic embedding. Let us denote the generators of $O(N)$ and Sp$(2M)$ as M_{ab} and $M_{\dot a\dot b}$, respectively. The invariant anti-symmetric tensor of Sp$(2M)$ is denoted by $\omega_{\dot a\dot b}$. We denote the bi-fundamental matter fields $\Phi^A_\alpha, \Psi^A_\alpha$ by

$$\Phi^{a\dot a}_\alpha,\quad \Psi^{a\dot a}_\alpha. \tag{2.15}$$

We choose the symplectic invariant tensor ω_{AB} as $\omega_{a\dot a,b\dot b} = \delta_{ab}\cdot\omega_{\dot a\dot b}$. The matter fields obey the reality condition of the form

$$\bar\Phi^\alpha_{\dot a a} \;=\; (\Phi^{a\dot a}_\alpha)^\dagger \;=\; \delta_{ab}\omega_{\dot a\dot b}C^{\alpha\beta}\Phi^{b\dot b}_\beta, \tag{2.16}$$

and similarly for the fermions. In the following the $O(N)$ vector indices are raised or lowered sloppily while the Sp$(2M)$ vector indices are raised or lowered by $\omega_{\dot a\dot b}$ and $\omega^{\dot a\dot b}$. Later we find it convenient to regard the matter fields as $N\times 2M$ matrices and omit the indices.

From the commutation relation of OSp($N|2M$) generators,

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \delta_{bc}M_{ad} - \delta_{bd}M_{ac} - \delta_{ac}M_{bd} + \delta_{ad}M_{bc}\,,\\
[M_{\dot a\dot b}, M_{\dot c\dot d}] &= \omega_{\dot b\dot c}M_{\dot a\dot d} + \omega_{\dot b\dot d}M_{\dot a\dot c} + \omega_{\dot a\dot c}M_{\dot b\dot d} + \omega_{\dot a\dot d}M_{\dot b\dot c}\,,\\
[M_{ab}, Q_{c\dot c}] &= \delta_{bc}Q_{a\dot c} - \delta_{ac}Q_{b\dot c}\,,\\
[M_{\dot a\dot b}, Q_{c\dot c}] &= \omega_{\dot a\dot c}Q_{c\dot b} + \omega_{\dot b\dot c}Q_{c\dot a}\,,\\
\{Q_{a\dot a}, Q_{b\dot b}\} &= \frac{k}{2\pi}(\omega_{\dot a\dot b}M_{ab} + \delta_{ab}M_{\dot a\dot b})\,,
\end{aligned} \tag{2.17}$$

one can read off the representation matrices on matters,

$$(t_{ab})_{c\dot c,d\dot d} = \omega_{\dot c\dot d}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\,,\qquad (t_{\dot a\dot b})_{c\dot c,d\dot d} = -\delta_{cd}(\omega_{\dot a\dot c}\omega_{\dot b\dot d} + \omega_{\dot a\dot d}\omega_{\dot b\dot c})\,, \tag{2.18}$$

and the quadratic invariant tensor (Chern-Simons coupling)

$$k^{ab,cd} \;=\; \frac{k}{8\pi}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})\,,\qquad k^{\dot a\dot b,\dot c\dot d} \;=\; -\frac{k}{8\pi}(\omega^{\dot a\dot c}\omega^{\dot b\dot d} + \omega^{\dot a\dot d}\omega^{\dot b\dot c})\,. \tag{2.19}$$

Lagrangian. The kinetic terms for matters are given by

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\text{tr}\left(-D_\mu\bar\Phi^\alpha D^\mu\Phi_\alpha + i\bar\Psi^\alpha\not{D}\Psi_\alpha\right)\,. \tag{2.20}$$

JHEP09(2008)002

We normalize the gauge fields for each gauge group $O(N)$ and $\mathrm{Sp}(2M)$ as

$$A_{O(N)} = \frac{1}{2} t_{ab} A^{ab} , \qquad A_{\mathrm{Sp}(2M)} = \frac{1}{2} t_{\dot a \dot b} (\omega^{\dot a \dot c} \tilde{A}_{\dot c}{}^{\dot b}) . \tag{2.21}$$

Then the Chern-Simons term becomes

$$\mathcal{L}_{\mathrm{CS}} = \frac{\epsilon^{\mu\nu\rho}}{4k} \mathrm{tr} \left(-A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho + \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) . \tag{2.22}$$

The Yukawa and bosonic potential terms are computed by substituting the following expressions into the currents and moment maps,

$$(\mathcal{J}_{ab})_{\alpha\beta} = (\Phi_\alpha \bar\Psi_\beta + \Psi_\beta \bar\Phi_\alpha)_{ab} , \qquad (\mathcal{J}_{\dot a \dot b})_{\alpha\beta} = (\bar\Phi_\alpha \Psi_\beta \omega + \bar\Psi_\beta \Phi_\alpha \omega)_{\dot a \dot b} , \tag{2.23}$$

$$(\mathcal{M}_{ab})_{\alpha\beta} = (\Phi_\alpha \bar\Phi_\beta + \Phi_\beta \bar\Phi_\alpha)_{ab} , \qquad (\mathcal{M}_{\dot a \dot b})_{\alpha\beta} = (\bar\Phi_\alpha \Phi_\beta \omega + \bar\Phi_\beta \Phi_\alpha \omega)_{\dot a \dot b} , \tag{2.24}$$

and so on.

For the computation of the interaction terms, it is useful to write the currents (2.23) and the moment maps (2.24) into the trace form,

$$(\mathcal{J}_{ab})_{\alpha\beta} = -\mathrm{tr}\left[\bar\Psi_\beta \tau_{ab} \Phi_\alpha\right] , \qquad (\mathcal{J}_{\dot a \dot b})_{\alpha\beta} = \mathrm{tr}\left[\bar\Psi_\beta \Phi_\alpha \tau_{\dot a \dot b}\right] , \tag{2.25}$$

$$(\mathcal{M}_{ab})_{\alpha\beta} = -\mathrm{tr}\left[\bar\Phi_\beta \tau_{ab} \Phi_\alpha\right] , \qquad (\mathcal{M}_{\dot a \dot b})_{\alpha\beta} = \mathrm{tr}\left[\bar\Phi_\beta \Phi_\alpha \tau_{\dot a \dot b}\right] , \tag{2.26}$$

Similarly, we also have

$$\begin{aligned} (\mathcal{M}_{ab,cd})_{\alpha\beta} &= +\mathrm{tr}\left[\bar\Phi_\beta \tau_{cd} \tau_{ab} \Phi_\alpha\right] , \\ (\mathcal{M}_{ab,\dot a \dot b})_{\alpha\beta} &= -\mathrm{tr}\left[\bar\Phi_\beta \tau_{ab} \Phi_\alpha \tau_{\dot a \dot b}\right] , \\ (\mathcal{M}_{\dot a \dot b, \dot c \dot d})_{\alpha\beta} &= +\mathrm{tr}\left[\bar\Phi_\beta \Phi_\alpha \tau_{\dot a \dot b} \tau_{\dot c \dot d}\right] . \end{aligned} \tag{2.27}$$

Here τ_{ab} and $\tau_{\dot a \dot b}$ are the matrices in the defining representation,

$$(\tau_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \qquad (\tau_{\dot a \dot b})_{\dot c}{}^{\dot d} = -\omega_{\dot a \dot c} \delta^{\dot d}_{\dot b} - \omega_{\dot b \dot c} \delta^{\dot d}_{\dot a}. \tag{2.28}$$

Using the completeness relations, we can rewrite the product of traces into a single trace,

$$\begin{aligned} k^{ab,cd} \mathrm{tr}\left[X \tau_{ab}\right] \mathrm{tr}\left[Y \tau_{cd}\right] &= -\frac{k}{\pi} \mathrm{tr}\left[X_- Y_-\right] , \\ k^{\dot a \dot b, \dot c \dot d} \mathrm{tr}\left[X \tau_{\dot a \dot b}\right] \mathrm{tr}\left[Y \tau_{\dot c \dot d}\right] &= +\frac{k}{\pi} \mathrm{tr}\left[X_+ Y_+\right] , \end{aligned} \tag{2.29}$$

where X_- and X_+ are the projections of X satisfying $X^T = -X$ and $(X\omega)^T = +X\omega$.

Some straightforward computations give the "Yukawa coupling",[1]

$$\begin{aligned} \mathcal{L}_{\mathrm{Yukawa}} = &+\frac{ik}{2} \mathrm{tr}\left[-\bar\Psi_\beta \Phi_\alpha \bar\Phi^\alpha \Psi^\beta + \Psi_\beta \bar\Phi_\alpha \Phi^\alpha \bar\Psi^\beta + 2 \bar\Psi_\alpha \Phi_\beta \bar\Phi^\alpha \Psi^\beta - 2 \Psi^\beta \bar\Phi^\alpha \Phi_\beta \bar\Psi_\alpha \right] \\ &- ik \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr}\left[\Phi_\alpha \bar\Psi_\beta \Phi_\gamma \bar\Psi_\delta\right] , \end{aligned} \tag{2.30}$$

[1] We are slightly abusing the term "Yukawa" to denote (2 fermion + 2 boson) quartic couplings.

JHEP09(2008)002

and the bosonic potential,

$$
\begin{aligned}
-V \; = \; & \frac{k^2}{6}\mathrm{tr}\{\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\bar{\Phi}^\alpha + \Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\gamma\bar{\Phi}^\gamma \\
& \qquad + 4\Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma\bar{\Phi}^\beta\Phi_\alpha\bar{\Phi}^\gamma - 6\Phi_\gamma\bar{\Phi}^\gamma\Phi_\beta\bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\} \; . \qquad (2.31)
\end{aligned}
$$

In the computation of these we used the Sp(4) identities

$$
\epsilon^{\alpha\beta\gamma\delta} \; = C^{\alpha\beta}C^{\gamma\delta} + C^{\alpha\gamma}C^{\delta\beta} + C^{\alpha\delta}C^{\beta\gamma} \, , \qquad (2.32)
$$

$$
\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\rho\sigma\tau} \; = 6\delta_\rho^{[\beta}\delta_\sigma^\gamma\delta_\tau^{\delta]} \; = \; -3\left(\delta_\rho^\beta C^{\gamma\delta}C_{\sigma\tau} + \delta_\rho^\gamma C^{\delta\beta}C_{\sigma\tau} + \delta_\rho^\delta C^{\beta\gamma}C_{\sigma\tau}\right) \, , \qquad (2.33)
$$

and an equality which follows directly from (2.33),

$$
\begin{aligned}
0 = \mathrm{tr}\{ & -\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\bar{\Phi}^\alpha - \Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\gamma\bar{\Phi}^\gamma - \Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma\bar{\Phi}^\beta\Phi_\alpha\bar{\Phi}^\gamma + 3\Phi_\gamma\bar{\Phi}^\gamma\Phi_\beta\bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta \\
& +3\Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}_\gamma\Phi^\beta\bar{\Phi}^\gamma - 3\bar{\Phi}_\alpha\Phi^\alpha\bar{\Phi}_\gamma\Phi_\beta\bar{\Phi}^\gamma\Phi^\beta + 3\Phi^\alpha\bar{\Phi}^\beta\Phi^\gamma\bar{\Phi}_\alpha\Phi_\gamma\bar{\Phi}_\beta\} \; . \qquad (2.34)
\end{aligned}
$$

In summary, the full Lagrangian for the $O(N) \times \mathrm{Sp}(2M)$ theory is

$$
\begin{aligned}
\mathcal{L} = \; & \frac{\epsilon^{\mu\nu\rho}}{4k}\mathrm{tr}\left(-A_\mu\partial_\nu A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho + \tilde{A}_\mu\partial_\nu\tilde{A}_\rho + \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\right) \\
& +\frac{1}{2}\mathrm{tr}\left(-D_\mu\bar{\Phi}^\alpha D^\mu\Phi_\alpha + i\bar{\Psi}^\alpha \not{D}\Psi_\alpha\right) - ik\epsilon^{\alpha\beta\gamma\delta}\mathrm{Tr}\left(\Phi_\alpha\bar{\Psi}_\beta\Phi_\gamma\bar{\Psi}_\delta\right) \\
& +\frac{ik}{2}\mathrm{tr}\left(-\bar{\Psi}_\beta\Phi_\alpha\bar{\Phi}^\alpha\Psi^\beta + \Psi_\beta\bar{\Phi}_\alpha\Phi^\alpha\bar{\Psi}^\beta + 2\bar{\Psi}_\alpha\Phi_\beta\bar{\Phi}^\alpha\Psi^\beta - 2\Psi^\beta\bar{\Phi}^\alpha\Phi_\beta\bar{\Psi}_\alpha\right) \\
& +\frac{k^2}{6}\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\bar{\Phi}^\alpha + \Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\gamma\bar{\Phi}^\gamma\right. \\
& \qquad \left. +4\Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma\bar{\Phi}^\beta\Phi_\alpha\bar{\Phi}^\gamma - 6\Phi_\gamma\bar{\Phi}^\gamma\Phi_\beta\bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\right) \; . \qquad (2.35)
\end{aligned}
$$

Supersymmetry transformation rules. The $\mathcal{N} = 5$ supersymmetry transformation rule for $\mathrm{OSp}(N|2M)$ model is given by

$$
\begin{aligned}
& \delta\Phi_\alpha = i\eta_\alpha{}^\beta\Psi_\beta \, , \\
& \delta A_\mu = \frac{ik}{2}\eta^{\alpha\beta}\gamma_\mu(\Phi_\alpha\bar{\Psi}_\beta + \Psi_\beta\bar{\Phi}_\alpha) \, , \quad \delta\tilde{A}_\mu = \frac{ik}{2}\eta^{\alpha\beta}\gamma_\mu(\bar{\Phi}_\alpha\Psi_\beta + \bar{\Psi}_\beta\Phi_\alpha) \, , \\
& \delta\Psi_\alpha = \not{D}\Phi_\gamma\eta^\gamma{}_\alpha + \frac{2k}{3}(\Phi_{[\gamma}\bar{\Phi}^\beta\Phi_{\beta]} + \Phi^\beta\bar{\Phi}_\gamma\Phi_\beta)\eta^\gamma{}_\alpha - \frac{4k}{3}(\Phi_{[\alpha}\bar{\Phi}^\gamma\Phi_{\beta]} + \Phi^\gamma\bar{\Phi}_\alpha\Phi_\beta)\eta^\beta{}_\gamma . \; (2.36)
\end{aligned}
$$

3. $\mathcal{N} = 6$ superconformal theories

In general, the Gaiotto-Witten construction we reviewed in section 2 assumes that the matter fields form a pseudo-real representation ($\mathcal{R}$) of the gauge group; see the reality conditions (2.1). If $\mathcal{R}$ can be decomposed into a complex representation (R) and its complex-conjugate representation ($\bar{R}$), then the $\mathcal{N} = 5$ supersymmetry is further enhanced to $\mathcal{N} = 6$.

3.1 General construction

The construction is an exercise of embedding the R-symmetry group SO(5) = Sp(4) into SU(4) = SO(6). The $\mathcal{N} = 5$ fields are decomposed into $\mathcal{N} = 6$ fields as[2]

$$(\Phi^A_\alpha)_{\mathcal{N}=5} = \begin{pmatrix} \Phi^A_\alpha \\ C_{\alpha\beta}\bar{\Phi}^\beta_A \end{pmatrix}, \qquad (\Psi^A_\alpha)_{\mathcal{N}=5} = \begin{pmatrix} C_{\alpha\beta}\Psi^{\beta A} \\ -\bar{\Psi}_{\alpha A} \end{pmatrix}. \tag{3.1}$$

With the symplectic invariant tensor,

$$(\omega_{AB})_{\mathcal{N}=5} = \begin{pmatrix} 0 & \delta_A{}^B \\ -\delta^A{}_B & 0 \end{pmatrix}, \tag{3.2}$$

the reality conditions reduce to

$$(\Phi^A_\alpha)^\dagger = \bar{\Phi}^\alpha_A, \qquad (\Psi^{\alpha A})^\dagger = \bar{\Psi}_{\alpha A}, \tag{3.3}$$

which is consistent with the following assignments we need for the lift to $\mathcal{N} = 6$.

$$\begin{array}{c|cccc} & \Phi^A_\alpha & \bar{\Phi}^\alpha_A & \Psi^{\alpha A} & \bar{\Psi}_{\alpha A} \\ \hline \text{Gauge} & R & \bar{R} & R & \bar{R} \\ \text{SO}(6)_R & \mathbf{4} & \bar{\mathbf{4}} & \bar{\mathbf{4}} & \mathbf{4} \end{array} \tag{3.4}$$

The gauge generators are written in a block diagonal form as

$$(t^A{}_B)_{\mathcal{N}=5} = \begin{pmatrix} t^A{}_B & 0 \\ 0 & -t^B{}_A \end{pmatrix}, \quad (t_{AB})_{\mathcal{N}=5} = -\begin{pmatrix} 0 & t^B{}_A \\ t^A{}_B & 0 \end{pmatrix}, \tag{3.5}$$

and the fundamental identity in the $\mathcal{N} = 6$ notation reads

$$(t^m)^A_{\ B}(t_m)^C_{\ D} + (t^m)^A_{\ D}(t_m)^C_{\ B} = 0\,. \tag{3.6}$$

The "moment map" and "current" operators have the decomposition,

$$(\mathcal{M}^m)^\alpha_{\ \beta} = -(M^m)^\alpha_{\ \beta} - C^{\alpha\delta}C_{\beta\gamma}(M^m)^\gamma_{\ \delta}\,, \tag{3.7}$$

$$(\mathcal{M}^{mn})^\alpha_{\ \beta} = -(M^{mn})^\alpha_{\ \beta} + C^{\alpha\delta}C_{\beta\gamma}(M^{nm})^\gamma_{\ \delta}\,, \tag{3.8}$$

$$(\mathcal{J}^m)_{\alpha\beta} = (J^m)_{\alpha\beta} - C_{\alpha\gamma}C_{\beta\delta}(\bar{J}^m)^{\gamma\delta}\,, \tag{3.9}$$

where we introduced the $\mathcal{N} = 6$ covariant quantities

$$(M^m)^\alpha_{\ \beta} \equiv \bar{\Phi}^\alpha_A(t^m)^A_{\ B}\Phi^B_\beta, \qquad (M^{mn})^\alpha_{\ \beta} \equiv \bar{\Phi}^\alpha_A(t^m t^n)^A_{\ B}\Phi^B_\beta\,, \tag{3.10}$$

$$(J^m)_{\alpha\beta} \equiv \Phi^B_\alpha(t^m)^A_{\ B}\bar{\Psi}_{\beta A}, \qquad (\bar{J}^m)^{\alpha\beta} \equiv \bar{\Phi}^\alpha_A(t^m)^A_{\ B}\Psi^{\beta B}. \tag{3.11}$$

To rewrite the $\mathcal{N} = 5$ Lagrangian of the previous section in an $\mathcal{N} = 6$ covariant form, we have to make sure that all references to $C_{\alpha\beta}$ disappear. Using the Sp(4) identity (2.32)

$$C^{\alpha\beta}C^{\gamma\delta} + C^{\alpha\gamma}C^{\delta\beta} + C^{\alpha\delta}C^{\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta},$$

[2] To avoid introducing new set of indices in every page, we are recycling not only the α, β indices, but also the A, B indices. They run from 1 to $2n$ in $\mathcal{N} = 5$ formulas, but 1 to n in $\mathcal{N} = 6$ formulas. Hopefully, the context would make it clear which notation is being used.

we can remove all $C_{\alpha\beta}$ at the expense of introducing $\epsilon_{\alpha\beta\gamma\delta}$ which survives the lift. After a slightly lengthy algebra (see appendix A), we obtain the $\mathcal{N}=6$ lift of the $\mathcal{N}=5$ Lagrangian,

$$\begin{aligned}\mathcal{L} = &\frac{\varepsilon^{\mu\nu\lambda}}{4\pi}\left(k_{mn}A_\mu^m\partial_\nu A_\lambda^n+\frac{1}{3}f_{mnp}A_\mu^mA_\nu^nA_\lambda^p\right)-D\bar{\Phi}^\alpha_AD\Phi^A_\alpha+i\bar{\Psi}_{\alpha A}\not{D}\Psi^{\alpha A}\\ &+i\pi\left[2(\bar{J}_m)^{\alpha\beta}(J^m)_{\alpha\beta}-4(\bar{J}_m)^{\alpha\beta}(J^m)_{\beta\alpha}+\epsilon^{\alpha\beta\gamma\delta}(J_m)_{\alpha\beta}(J^m)_{\gamma\delta}+\epsilon_{\alpha\beta\gamma\delta}(\bar{J}_m)^{\alpha\beta}(\bar{J}^m)^{\gamma\delta}\right]\\ &-\frac{4\pi^2}{3}f_{mnp}(M^m)^\alpha{}_\beta(M^n)^\beta{}_\gamma(M^p)^\gamma{}_\alpha+4\pi^2(M^{mn})^\alpha{}_\beta(M_m)^\beta{}_\gamma(M_n)^\gamma{}_\alpha. \qquad (3.12)\end{aligned}$$

and the supersymmetry transformation law,

$$\begin{aligned}\delta\Phi^A_\alpha &= -i\eta_{\alpha\beta}\Psi^{A\beta}, \quad \delta A_\mu^m = 2\pi i\left(\eta^{\alpha\beta}\gamma_\mu(J^m)_{\alpha\beta}+\eta_{\alpha\beta}\gamma_\mu(\bar{J}^m)^{\alpha\beta}\right),\\ \delta\Psi^{A\alpha} &= \left[\not{D}\Phi^A_\gamma-\frac{2\pi}{3}(t_m)^A{}_B\Phi^B_\beta(M^m)^\beta{}_\gamma\right]\eta^{\gamma\alpha}+\frac{4\pi}{3}(t_m)^A{}_B\Phi^B_\beta(M^m)^\alpha{}_\gamma\eta^{\gamma\beta}\\ &\quad-\frac{2\pi}{3}\epsilon^{\alpha\beta\gamma\delta}(t_m)^A{}_B\Phi^B_\beta(M^m)^\rho{}_\gamma\eta_{\delta\rho}\,. \qquad (3.13)\end{aligned}$$

The parameter $\eta_{\alpha\beta}$ satisfies

$$\eta_{\alpha\beta}=-\eta_{\beta\alpha}\,,\qquad (\eta^*)^{\alpha\beta}=\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\eta_{\gamma\delta}\,. \qquad (3.14)$$

3.2 $\mathrm{U}(M|N)$ example

Symplectic embedding. Let us denote the $\mathrm{U}(M)$ and $\mathrm{U}(N)$ generators as $M_{a\underline{a}}$ and $M_{\dot{a}\underline{\dot{a}}}$, respectively. Here the indices without underlines indicate fundamental representation while those with underlines indicate anti-fundamental representation. For the present model, the complex matter fields Φ^A_α and $\Psi^{A\alpha}$ are described as

$$\Phi^A_\alpha=(\Phi_\alpha)^{a\underline{\dot{a}}}\,,\qquad \Psi^{A\alpha}=(\Psi^\alpha)^{a\underline{\dot{a}}}\,, \qquad (3.15)$$

and their complex conjugate fields as

$$\bar{\Phi}^\alpha_A=(\bar{\Phi}^\alpha)^{\dot{a}\underline{a}}\,,\qquad \bar{\Psi}_{A\alpha}=(\bar{\Psi}_\alpha)^{\dot{a}\underline{a}}\,. \qquad (3.16)$$

Hereafter we omit the indices $a,\dot{a},\underline{a},\underline{\dot{a}}$ and regard Φ_α,Ψ^α as $M\times N$ matrices. We choose the symplectic invariant tensor as

$$\omega_{a\underline{\dot{a}},\dot{b}\underline{b}}=-\omega_{\dot{b}\underline{b},a\underline{\dot{a}}}=\delta_{a\underline{b}}\delta_{\dot{b}\underline{\dot{a}}}\,. \qquad (3.17)$$

From the commutation relation of the Lie super-algebra $\mathrm{U}(M|N)$

$$\begin{aligned}&[M_{a\underline{b}},Q_{c\underline{\dot{c}}}]=+\delta_{c\underline{b}}Q_{a\underline{\dot{c}}}, \quad [M_{\dot{a}\underline{\dot{b}}},Q_{c\underline{\dot{c}}}]=-\delta_{\dot{a}\underline{\dot{c}}}Q_{c\underline{\dot{b}}}, \quad [M_{a\underline{b}},M_{c\underline{d}}]=\delta_{c\underline{b}}M_{a\underline{d}}-\delta_{a\underline{d}}M_{c\underline{b}},\\ &[M_{a\underline{b}},\bar{Q}_{\dot{c}\underline{c}}]=-\delta_{a\underline{c}}\bar{Q}_{\dot{c}\underline{b}}, \quad [M_{\dot{a}\underline{\dot{b}}},\bar{Q}_{\dot{c}\underline{c}}]=+\delta_{\dot{c}\underline{\dot{b}}}\bar{Q}_{\dot{a}\underline{c}}, \quad [M_{\dot{a}\underline{\dot{b}}},M_{\dot{c}\underline{\dot{d}}}]=\delta_{\dot{c}\underline{\dot{b}}}M_{\dot{a}\underline{\dot{d}}}-\delta_{\dot{a}\underline{\dot{d}}}M_{\dot{c}\underline{\dot{b}}},\\ &\qquad\qquad \{Q_{a\underline{\dot{a}}},\bar{Q}_{\dot{b}\underline{b}}\}=\frac{k}{2\pi}\left(\delta_{\dot{b}\underline{\dot{a}}}M_{a\underline{b}}+\delta_{a\underline{b}}M_{\dot{b}\underline{\dot{a}}}\right), \qquad (3.18)\end{aligned}$$

one reads off the representation matrix of gauge group on matters

$$(t_{a\underline{b}})_{c\underline{\dot{c}},\dot{d}\underline{d}}=-\delta_{\dot{d}\underline{\dot{c}}}\delta_{c\underline{b}}\delta_{a\underline{d}}\,,\qquad (t_{\dot{a}\underline{\dot{b}}})_{c\underline{\dot{c}},\dot{d}\underline{d}}=\delta_{c\underline{d}}\delta_{\dot{a}\underline{\dot{c}}}\delta_{\dot{d}\underline{\dot{b}}}\,, \qquad (3.19)$$

and the quadratic invariant tensor

$$k^{a\underline{b},c\underline{d}}=-\frac{k}{2\pi}\delta^{a\underline{d}}\delta^{c\underline{b}}\,,\qquad k^{\dot{a}\underline{\dot{b}},\dot{c}\underline{\dot{d}}}=+\frac{k}{2\pi}\delta^{\dot{a}\underline{\dot{d}}}\delta^{\dot{c}\underline{\dot{b}}}\,. \qquad (3.20)$$

Lagrangian. Once we normalize gauge fields for each gauge group $\mathrm{U}(M)$ and $\mathrm{U}(N)$ as

$$A_{\mathrm{U}(M)} = t_{a\underline{b}} A^{a\underline{b}}, \qquad \tilde{A}_{\mathrm{U}(N)} = t_{\dot{a}\underline{\dot{b}}} \tilde{A}^{\dot{a}\underline{\dot{b}}}, \tag{3.21}$$

it is easy to write down the Chern-Simons term and the matter kinetic term in the matrix form. To express the remaining interactions in terms of matrix fields Φ and Ψ, it is useful to write the currents and the moment maps in the trace form,

$$\begin{aligned} (J_{a\underline{b}})_{\alpha\beta} &= \mathrm{tr}[\Phi_\alpha \bar{\Psi}_\beta \tau_{a\underline{b}}], & (J_{\dot{a}\underline{\dot{b}}})_{\alpha\beta} &= -\mathrm{tr}[\Phi_\alpha \tau_{\dot{a}\underline{\dot{b}}} \bar{\Psi}_\beta], \\ (\bar{J}_{a\underline{b}})^{\alpha\beta} &= \mathrm{tr}[\Psi^\beta \bar{\Phi}^\alpha \tau_{a\underline{b}}], & (\bar{J}_{\dot{a}\underline{\dot{b}}})^{\alpha\beta} &= -\mathrm{tr}[\Psi^\beta \tau_{\dot{a}\underline{\dot{b}}} \bar{\Phi}^\alpha], \\ (M_{a\underline{b}})^\alpha{}_\beta &= \mathrm{tr}[\Phi_\beta \bar{\Phi}^\alpha \tau_{a\underline{b}}], & (M_{\dot{a}\underline{\dot{b}}})^\alpha{}_\beta &= -\mathrm{tr}[\Phi_\beta \tau_{\dot{a}\underline{\dot{b}}} \bar{\Phi}^\alpha], \end{aligned} \tag{3.22}$$

$$\begin{aligned} (M_{a\underline{b},c\underline{d}})^\alpha{}_\beta &= +\mathrm{tr}[\Phi_\beta \bar{\Phi}^\alpha \tau_{a\underline{b}} \tau_{c\underline{d}}], \\ (M_{a\underline{b},\dot{a}\underline{\dot{b}}})^\alpha{}_\beta &= -\mathrm{tr}[\Phi_\beta \tau_{\dot{a}\underline{\dot{b}}} \bar{\Phi}^\alpha \tau_{a\underline{b}}], \\ (M_{\dot{a}\underline{\dot{b}},\dot{c}\underline{\dot{d}}})^\alpha{}_\beta &= +\mathrm{tr}[\Phi_\beta \tau_{\dot{c}\underline{\dot{d}}} \tau_{\dot{a}\underline{\dot{b}}} \bar{\Phi}^\alpha], \end{aligned} \tag{3.23}$$

The products of traces can be simplified using completeness relations

$$k^{a\underline{b},c\underline{d}} \mathrm{tr}[X\tau_{a\underline{b}}]\mathrm{tr}[Y\tau_{c\underline{d}}] = -\frac{k}{2\pi}\mathrm{tr}[XY], \qquad k^{\dot{a}\underline{\dot{b}},\dot{c}\underline{\dot{d}}} \mathrm{tr}[X\tau_{\dot{a}\underline{\dot{b}}}]\mathrm{tr}[Y\tau_{\dot{c}\underline{\dot{d}}}] = \frac{k}{2\pi}\mathrm{tr}[XY]. \tag{3.24}$$

To summarize, the full $\mathcal{N}=6$ supersymmetric Lagrangian for the $\mathrm{U}(M)\times\mathrm{U}(N)$ theory is

$$\begin{aligned} \mathcal{L} = &-\frac{\epsilon^{\mu\nu\rho}}{2k}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu\partial_\nu\tilde{A}_\rho - \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\right) \\ &-\mathrm{tr}\left(D_\mu\bar{\Phi}^\alpha D^\mu\Phi_\alpha - i\bar{\Psi}_\alpha\gamma^\mu D_\mu\Psi^\alpha\right) - ik\epsilon^{\alpha\beta\gamma\delta}\mathrm{tr}\left(\Phi_\alpha\bar{\Psi}_\beta\Phi_\gamma\bar{\Psi}_\delta\right) + ik\epsilon_{\alpha\beta\gamma\delta}\mathrm{tr}\left(\bar{\Phi}^\alpha\Psi^\beta\bar{\Phi}^\gamma\Psi^d\right) \\ &-ik\mathrm{tr}\left(\bar{\Phi}^\alpha\Phi_\alpha\bar{\Psi}_\beta\Psi^\beta - \Phi_\alpha\bar{\Phi}^\alpha\Psi^\beta\bar{\Psi}_\beta + 2\bar{\Phi}^\alpha\Psi^\beta\bar{\Psi}_\alpha\Phi_\beta - 2\Phi_\alpha\bar{\Psi}_\beta\Psi^\alpha\bar{\Phi}^\beta\right) \\ &+\frac{1}{3}k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\beta\Phi_\gamma\bar{\Phi}^\gamma + \bar{\Phi}^\alpha\Phi_\alpha\bar{\Phi}^\beta\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\right) + \frac{4}{3}k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\gamma\Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma\bar{\Phi}^\beta\right) \\ &-2k^2\mathrm{tr}\left(\Phi_\alpha\bar{\Phi}^\alpha\Phi_\beta\bar{\Phi}^\gamma\Phi_\gamma\bar{\Phi}^\beta\right) \ . \end{aligned} \tag{3.25}$$

The bosonic part is precisely that of the ABJM model [11] and the Yukawa term agrees with that obtained in ref. [27]. Note that the $\mathcal{N}=6$ Lagrangian above looks almost identical to the $\mathcal{N}=5$ Lagragian (2.35) of the $\mathrm{OSp}(N|2M)$ model, except for the reality condition (2.16) for the latter. It follows that the moduli space of vacua of the $\mathrm{OSp}(N|2M)$ theory should be that of the $\mathrm{U}(N|2M)$ theory modded out by the reality condition. We will come back to this point in section 4.

Supersymmetry transformation rules. For scalar and gauge fields, one find

$$\begin{aligned} &\delta\Phi_\alpha = -i\eta_{\alpha\beta}\Psi^\beta, \quad \delta A_\mu = ik(\eta^{\alpha\beta}\gamma_\mu\Phi_\alpha\bar{\Psi}_\beta + \eta_{\alpha\beta}\gamma_\mu\Psi^\beta\bar{\Phi}^\alpha), \\ &\delta\tilde{A}_\mu = ik(\eta^{\alpha\beta}\gamma_\mu\bar{\Psi}_\beta\Phi_\alpha + \eta_{\alpha\beta}\gamma_\mu\bar{\Phi}^\alpha\Psi^\beta) \ . \end{aligned} \tag{3.26}$$

The supersymmetry transformation rule for fermions now becomes

$$\delta\Psi^\alpha = \left[\gamma^\mu D_\mu\Phi_\gamma - \frac{2k}{3}(\Phi_{[\beta}\bar{\Phi}^\beta\Phi_{\gamma]})\right]\eta^{\gamma\alpha} + \frac{4k}{3}(\Phi_\beta\bar{\Phi}^\alpha\Phi_\gamma)\eta^{\gamma\beta} - \frac{2k}{3}\epsilon^{\alpha\beta\gamma\delta}(\Phi_\beta\bar{\Phi}^\rho\Phi_\gamma)\eta_{\delta\rho}\,, \tag{3.27}$$

in agreement with a recent independent work [38].

3.3 $\mathrm{OSp}(2|2M)$ example

We now describe new $\mathcal{N}=6$ superconformal Chern-Simons theories for the super-algebra $\mathrm{OSp}(2|2M)$. The commutation relations were already discussed in section 2.3.

Symplectic embedding. $\mathrm{U}(1)=\mathrm{SO}(2)$ and $\mathrm{Sp}(2M)$ generators are denoted by M_{+-} and M_{ab}, respectively. Matter fields Φ^A_α and $\Psi^{A\alpha}$ are denoted by

$$\Phi^A_\alpha = (\Phi_\alpha)^{+a}, \qquad \Psi^{A\alpha} = (\Psi^\alpha)^{+a}, \tag{3.28}$$

and their complex conjugate fields by

$$\bar\Phi^\alpha_A = (\bar\Phi^\alpha)_{+a}, \qquad \bar\Psi_{A\alpha} = (\bar\Psi_\alpha)_{+a} \ . \tag{3.29}$$

For clarity, we hereafter suppress the $\mathrm{U}(1)$ and symplectic indices of the matter fields. We choose the symplectic invariant tensor to be

$$\omega_{+a,-b} = \omega_{-a.+b} = \omega_{ab} \,, \tag{3.30}$$

and the representation of gauge group on matter fields to be

$$(t_{+-})_{-a,+b} = -(t_{+-})_{+a,-b} = \omega_{ab} \tag{3.31}$$

$$(t_{ab})_{\pm c,\mp d} = -(\omega_{ac}\omega_{bd}+\omega_{ad}\omega_{bc}) \ . \tag{3.32}$$

The canonical expression used in section 3.1 can be obtained by

$$\begin{aligned}(t_m)^A{}_B : \ (t_{+-})^{+a}_{\ +b} &= \omega^{+a,-c}(t_{+-})_{-c,+b} = \delta^a{}_b, \\ (t_{ab})^{+c}_{\ +d} &= \omega^{+c,-f}(t_{ab})_{-f,+d} = \delta^c{}_a\omega_{bd} + \delta^c{}_b\omega_{ad} \ .\end{aligned} \tag{3.33}$$

The quadratic invariant tensor for $\mathrm{OSp}(2|M)$ reads

$$k^{+-,+-} = -\frac{k}{2\pi}, \qquad k^{ab,cd} = -\frac{k}{8\pi}\left(\omega^{ac}\omega^{bd}+\omega^{ad}\omega^{bc}\right) \ . \tag{3.34}$$

Before closing this paragraph, let us present some useful quantities to be used below,

$$\begin{aligned}(t_{+-}t_{+-})_{\pm a,\mp b} &= \omega_{ab}, \\ (t_{ab}t_{cd})_{\pm f,\mp g} &= -\left(\omega_{af}\omega_{bc}\omega_{dg}+\omega_{bf}\omega_{ac}\omega_{dg}+\omega_{af}\omega_{bd}\omega_{cg}+\omega_{bf}\omega_{ad}\omega_{cg}\right) \\ (t_{+-}t_{ab})_{\pm c,\mp d} &= (t_{ab}t_{+-})_{\pm c,\mp d} = \pm\left(\omega_{ac}\omega_{bd}+\omega_{ad}\omega_{bc}\right) \ .\end{aligned} \tag{3.35}$$

Lagrangian. In our convention, we obtain the kinetic terms for matter fields

$$\mathcal{L}_{\rm kin} = -D^\mu\Phi_\alpha D_\mu\bar\Phi^\alpha + i\Psi^\alpha\gamma^\mu D_\mu\bar\Psi_\alpha \,, \tag{3.36}$$

together with the Chern-Simons term

$$\mathcal{L}_{\rm CS} = -\frac{\epsilon^{\mu\nu\rho}}{2k}A_\mu\partial_\nu A_\rho + \frac{\epsilon^{\mu\nu\rho}}{4k}\mathrm{tr}\left(\tilde A_\mu\partial_\nu\tilde A_\rho + \frac{2}{3}\tilde A_\mu\tilde A_\nu\tilde A_\rho\right) \ . \tag{3.37}$$

JHEP09(2008)002

Here we normalized the gauge fields as

$$A_{\mathrm{U}(1)} = t_{+-}A, \quad \tilde{A}_{\mathrm{Sp}(2M)} = \frac{1}{2}t_{ab}\tilde{A}^{ab}, \tag{3.38}$$

As usual, explicit expressions of the Yukawa interactions and scalar potentials can be easily computed once we substitute the moment map and current operators for the present model,

$$(M_{+-})^{\dot\alpha}{}_{\dot\beta} = +(\Phi_\beta\bar{\Phi}^\alpha), \quad (M_{ab})^\alpha{}_\beta = +(\bar{\Phi}^\alpha\Phi_\beta)_{ab} + (\bar{\Phi}^\alpha\Phi_\beta)_{ba}\,, \tag{3.39}$$

and

$$\begin{aligned}(J_{+-})_{\alpha\beta} &= +(\Phi_\alpha\bar{\Psi}_\beta), & (J_{ab})_{\alpha\beta} &= +(\bar{\Psi}_\beta\Phi_\alpha)_{ab} + (\bar{\Psi}_\beta\Phi_\alpha)_{ba},\\ (\bar{J}_{+-})^{\alpha\beta} &= +(\Psi^\beta\bar{\Phi}^\alpha)\,, & (\bar{J}_{ab})^{\alpha\beta} &= +(\bar{\Phi}^\alpha\Psi^\beta)_{ab} + (\bar{\Phi}^\alpha\Psi^\beta)_{ba}\,.\end{aligned} \tag{3.40}$$

As for the symplectic summation convention, we take $\Phi_1\bar{\Phi}_2 \equiv \Phi_1^a\bar{\Phi}_{2a}$.

In summary, the full Lagrangian is $\mathcal{L} = \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{CS}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{potential}}$, where

$$\begin{aligned}\mathcal{L}_{\mathrm{kin}} &= -D^\mu\Phi_\alpha D_\mu\bar{\Phi}^\alpha + i\Psi^\alpha\gamma^\mu D_\mu\bar{\Psi}_\alpha,\\ \mathcal{L}_{\mathrm{CS}} &= -\frac{\epsilon^{\mu\nu\rho}}{2k}A_\mu\partial_\nu A_\rho + \frac{\epsilon^{\mu\nu\rho}}{4k}\mathrm{tr}\left(\tilde{A}_\mu\partial_\nu\tilde{A}_\rho + \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\right),\\ \mathcal{L}_{\mathrm{Yukawa}} &= -ik\left(\Phi_\alpha\bar{\Psi}_\beta\cdot\Psi^\beta\bar{\Phi}^\alpha - \Phi_\alpha\bar{\Phi}^\alpha\cdot\Psi^\beta\bar{\Psi}_\beta - \Phi_\alpha\omega\Psi^\beta\cdot\bar{\Psi}_\beta\omega\bar{\Phi}^\alpha\right)\\ &\quad +2ik\left(\Phi_\alpha\bar{\Psi}_\beta\cdot\Psi^\alpha\bar{\Phi}^\beta - \Phi_\alpha\bar{\Phi}^\beta\cdot\Psi^\alpha\bar{\Psi}_\beta - \Phi_\alpha\omega\Psi^\alpha\cdot\bar{\Psi}_\beta\omega\bar{\Phi}^\beta\right)\\ &\quad -ik\epsilon^{\alpha\beta\gamma\delta}\left(\Phi_\alpha\bar{\Psi}_\beta\cdot\Phi_\gamma\bar{\Psi}_\delta - \frac{1}{2}\Phi_\alpha\omega\Phi_\gamma\cdot\bar{\Psi}_\beta\omega\bar{\Psi}_\delta\right)\\ &\quad -ik\epsilon_{\alpha\beta\gamma\delta}\left(\Psi^\beta\bar{\Phi}^\alpha\cdot\Psi^\delta\bar{\Phi}^\gamma - \frac{1}{2}\bar{\Phi}^\alpha\omega\bar{\Phi}^\gamma\cdot\Psi^\beta\omega\Psi^\delta\right),\\ \mathcal{L}_{\mathrm{potential}} &= -3k^2\left(\bar{\Phi}^\alpha\omega\bar{\Phi}^\beta\cdot\Phi_\beta\bar{\Phi}^\gamma\cdot\Phi_\gamma\omega\Phi_\alpha\right) + \frac{5k^2}{3}\left(\Phi_\alpha\bar{\Phi}^\beta\cdot\Phi_\beta\bar{\Phi}^\gamma\cdot\Phi_\gamma\bar{\Phi}^\alpha\right)\\ &\quad -2k^2\left(\Phi_\alpha\bar{\Phi}^\gamma\cdot\Phi_\beta\bar{\Phi}^\beta\cdot\Phi_\gamma\bar{\Phi}^\alpha\right) + \frac{k^2}{3}\left(\Phi_\alpha\bar{\Phi}^\alpha\cdot\Phi_\beta\bar{\Phi}^\beta\cdot\Phi_\gamma\bar{\Phi}^\gamma\right)\,.\end{aligned} \tag{3.41}$$

Here we used the notations

$$\Phi_1\omega\Phi_2 = \Phi_1^a\omega_{ab}\Phi_2^b, \quad \bar{\Phi}_1\omega\bar{\Phi}_2 = \bar{\Phi}_{1a}\omega^{ab}\bar{\Phi}_{2b}\,. \tag{3.42}$$

Supersymmetry transformation rules. The $\mathcal{N} = 6$ supersymmetry transformation rules for $\mathrm{OSp}(2|M)$ model are given by

$$\begin{aligned}&\delta\Phi_\alpha = -i\eta_{\alpha\beta}\Psi^\beta, \quad \delta A_\mu = -ik(\eta^{\alpha\beta}\gamma_\mu\Phi_\alpha\bar{\Psi}_\beta + \eta_{\alpha\beta}\gamma_\mu\Psi^\beta\bar{\Phi}^\alpha),\\ &\delta A_\mu^{ab} = -ik(\eta^{\alpha\beta}\gamma_\mu\bar{\Psi}_\beta\Phi_\alpha + \eta_{\alpha\beta}\gamma_\mu\bar{\Phi}^\alpha\Psi^\beta)^{(ab)}\,.\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\delta\Psi^\alpha = &\left(\gamma^\mu D_\mu\Phi_\gamma - \frac{2k}{3}\Phi_{[\beta}\bar{\Phi}^\beta\cdot\Phi_{\gamma]} + \frac{k}{3}\omega\bar{\Phi}^\beta\cdot\Phi_\beta\omega\Phi_\gamma\right)\eta^{\gamma\alpha}\\ &+\left(\frac{4k}{3}\Phi_\beta\bar{\Phi}^\alpha\cdot\Phi_\gamma - \frac{2k}{3}\omega\bar{\Phi}^\alpha\cdot\Phi_\beta\omega\Phi_\gamma\right)\eta^{\gamma\beta}\\ &+\epsilon^{\alpha\beta\gamma\delta}\left(\frac{2k}{3}\Phi_\beta\bar{\Phi}^\rho\cdot\Phi_\gamma - \frac{k}{3}\omega\bar{\Phi}^\rho\cdot\Phi_\beta\omega\Phi_\gamma\right)\eta_{\delta\rho}\,.\end{aligned} \tag{3.44}$$

4. IIB orientifold and M2-branes on orbifold

In [11] it was argued that the $\mathcal{N}=6$ $\mathrm{U}(N)\times\mathrm{U}(N)$ ABJM model with CS coupling k is the world-volume theory of N M2-branes in orbifold $\mathbb{C}^4/\mathbb{Z}_k$.[3] Here we argue that our $\mathcal{N}=5$ theory with the gauge group $\mathrm{SO}(2N)\times\mathrm{Sp}(2N)$ and the CS coupling $2k$ is the world-volume theory of N M2-branes in orbifold $\mathbb{C}^4/\hat{D}_{k+2}$, where $\hat{D}_{k+2}$ is the binary dihedral group with $4k$ elements.

Our arguments closely follow that of [11]. We take the orientifold of a Type IIB brane configuration realizing the ABJM model, and consider its M-theory dual. We show how the orientifold breaks supersymmetry down to $\mathcal{N}=5$ from the viewpoint of M-theory geometry as well as the world-volume field theory.

Brane construction of ABJM theory. The ABJM model with gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ can be embedded in IIB superstring theory in flat spacetime with compact x_6 direction. Consider N D3-branes(0126) intersecting with an NS5-brane(012345) and an $(1,k)$ 5-brane(0123'4'5') at different points on the $S^1(x_6)$. The directions $3',4',5'$ are given by rotating $3,4,5$ by the same angle θ in the planes $37, 48$ and 59 respectively. The D3-brane world-volume theory is a $\mathcal{N}=3$ $\mathrm{U}(N)\times\mathrm{U}(N)$ Yang-Mills Chern-Simons theory which flows to the ABJM model in the IR limit.

T-duality along the x_6 direction followed by an M-theory lift gives a theory of N M2-branes. The transverse space is a fibration of $T^2(\tilde{x}_6,x_{10})$ over $\mathbb{R}^6(x_3,x_4,x_5,x_{3'},x_{4'},x_{5'})$. The 5-branes turn into Taub-NUT type geometry after the duality chain. The M-theory geometry is given by a $\mathbb{Z}_k$ orbifold of the product of two Taub-NUTs, where the $\mathbb{Z}_k$ is the simultaneous translation along the two S^1 fibers by $1/k$-period. Thus the ABJM model describes N M2-branes at the orbifold $\mathbb{C}^4/\mathbb{Z}_k$.

The four bi-fundamental scalars in ABJM model, which we denote by $(\phi_1,\phi_2,\phi_3,\phi_4)$ here, are identified with complex coordinates of the orbifold. As an example, for $N=1$ the scalars (ϕ_i) are complex numbers. The $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge fields removes one dimension of the moduli space through gauge equivalence and adds one back through the dual photon. The net effect is the $\mathbb{Z}_k$ orbifolding.

$$\alpha\ :\ \phi_i\ \longrightarrow\ e^{2\pi i/k}\phi_i. \tag{4.1}$$

The SU(4) R-symmetry of ABJM model has a geometric interpretation as the subgroup of transverse SO(8) rotations which commutes with the $\mathbb{Z}_k$ orbifolding.

Introduction of orientifold. Back in Type IIB setup, introducing the O3-plane on top of $2N$ D3-branes with one NS 5-brane and one (1,k) 5-brane gives an $\mathrm{SO}(2N)\times\mathrm{Sp}(2N)$ gauge theory [39]. The type and the charge of the O3-plane change across either type of 5-brane so that we have the gauge group $\mathrm{SO}(2N)\times\mathrm{Sp}(2N)$. This is explained in detail [40]. Following the chain of duality to M-theory, one finds that the orientifold turns into an orbifold (β) that flips all the coordinates $(3,4,5;3',4',5';\tilde{6},10)$. Since the directions $\tilde{6}$ and

[3] To avoid confusion, note that the letter "k" used in earlier sections (2.3, 3.2, 3.3) of this paper is inversely related to the integer-quantized Chern-Simons level k of this section.

JHEP09(2008)002

10 make the phase directions of the fields ϕ_i and β reverses them, β should act anti-holomorphically on these fields. Also, if one requires that the origin is the only fixed point under β, the action cannot be involutive.

Let us recall the simpler system of O6$^-$-plane and k D6-branes that uplifts to the M-theory on $\hat{D}_{k+2}$ orbifold. The generators α, β of the orbifold group $\hat{D}_{k+2}$ correspond to the $1/2k$-period shift along the M-theory circle and the orientifold, respectively. They satisfy

$$\alpha^{2k} = 1, \qquad \beta^2 = \alpha^k, \qquad \beta\alpha\beta^{-1} = \alpha^{-1}. \tag{4.2}$$

So β squares to the half-period shift along the M-theory circle. If the same rule applies to our case, then β^2 should flip the sign of all the fields ϕ_i. From anti-holomorphicity and $\beta^2 = -1$, the action of β on fields should be of this form

$$\beta \ : \ (\phi_1, \phi_2, \phi_3, \phi_4) \ \longrightarrow \ (\phi_2^*, -\phi_1^*, \phi_4^*, -\phi_3^*), \tag{4.3}$$

up to a linear redefinition of fields. This is equivalent to the reality condition of matter fields in $\mathcal{N} = 5$ supersymmetric theory of section 2.2 involving the matrix $C_{\alpha\beta}$. This new orbifold element breaks the transverse rotation symmetry further to $\mathrm{Sp}(4) \simeq \mathrm{SO}(5)$, in consistency with the supersymmetry of the world-volume theory.

Orientifolding the field theory. As noted in ref. [11], the ABJM model written in $d = 3$, $\mathcal{N} = 2$ super-field notation closely resembles the conifold theory [41] in $d = 4$, $\mathcal{N} = 1$ notation. The reason is that both theories have dual descriptions in terms of D-branes winding around a circle and intersecting with two 5-branes at different points on the circle. Also, the two 5-branes are tilted relative to each other in both theories, albeit in somewhat different ways.

Here we argue that the orientifold action (4.3) can be obtained by the standard open string analysis. The bi-fundamental matter fields of ABJM model arise from the open string connecting a pair of D-branes separated by a 5-brane. The orientifold projection on those fields should be independent of the relative angle of the two 5-branes.

We start with the conifold theory described as $\mathrm{U}(2N) \times \mathrm{U}(2N)$ gauge theory with two bi-fundamental fields A_i $(2N, \overline{2N})$ and B_i $(\overline{2N}, 2N)$ and the super-potential

$$W = \mathrm{Tr}(A_1 B_2 A_2 B_1 - A_1 B_1 A_2 B_2). \tag{4.4}$$

One encounters the same super-potential when writing the ABJM model in $d = 3, \mathcal{N} = 2$ chiral super-fields A_i, B_i. Their lowest components are combined into an SU(4) multiplet,

$$\Phi = \begin{pmatrix} A_i \\ B_i^\dagger \end{pmatrix}, \qquad \bar{\Phi} = \begin{pmatrix} (A^\dagger)^i \\ B^i \end{pmatrix}. \tag{4.5}$$

The orientifold of the conifold model relevant for our discussion was discussed in section 4.3.2 of [42]. This is the only orientifold model which gives the right gauge group $\mathrm{SO}(2N) \times \mathrm{Sp}(2N)$ to match the N=5 theory of our interest. There it was shown that the orientifold acts on the matter fields as the $\mathbb{Z}_2$ identification

$$A_1 = B_2^T J \equiv A, \ \ A_2 = -B_1^T J \equiv B. \tag{4.6}$$

where J is a matrix form of the anti-symmetric invariant tensor of $\mathrm{Sp}(2N)$ satisfying $J^2 = -1$. The reason why we should impose the condition $J^2 = -1$ instead of $J^2 = 1$ on the Chan-Paton factors is explained at [43]. The resulting theory is an $\mathrm{SO}(2N) \times \mathrm{Sp}(2N)$ gauge theory with two bi-fundamental fields A, B and the super-potential

$$W = \mathrm{Tr}(AB^TBA^T - BB^TAA^T). \tag{4.7}$$

One can see that (4.6) is nothing but the reality condition on the field Φ of section 2.2 up to a trivial change of basis.

Acknowledgments

We thank Nakwoo Kim and David Tong and for discussions. K.M.L. and J.P. are supported in part by the KOSEF SRC Program through CQUeST at Sogang University. K.M.L. is supported in part by KRF Grant No. KRF-2005-070-C00030, and the KRF National Scholar program. Sm.L. is supported in part by the KOSEF Grant R01-2006-000-10965-0 and the Korea Research Foundation Grant KRF-2007-331-C00073. K.H. acknowledges the hospitality of CEA Saclay where parts of the work were carried out. Sm.L. acknowledges the hospitality of Simons Workshop 2008 where the paper was finalized. J.P. is supported in part by the Stanford Institute for Theoretical Physics. J.P. acknowledges the hospitality of the Aspen Center for Physics where parts of the project were done.

A. Details of computation

A.1 The $\mathcal{N} = 5$ case

Yukawa term. We start from the "Yukawa terms" of $\mathcal{N} = 4$ Lagrangian (2.4). By applying the fundamental identity for t^m to the last two terms we get,

$$\begin{aligned}
\mathcal{L}_{\mathrm{Y}} = i\pi\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} \Big\{ &-(q_\alpha t^m \psi_\gamma)(q_\beta t_m \psi_\delta) - (\tilde{q}_\alpha t^m \tilde{\psi}_\gamma)(\tilde{q}_\beta t_m \tilde{\psi}_\delta) + 4(q_\alpha t^m \psi_\gamma)(\tilde{q}_\delta t_m \tilde{\psi}_\beta) \\
&- (q_\alpha t^m \tilde{\psi}_\beta)(q_\gamma t_m \tilde{\psi}_\delta) - (\tilde{q}_\alpha t^m \psi_\beta)(\tilde{q}_\gamma t_m \psi_\delta) \\
&- (q_\alpha t^m \tilde{\psi}_\delta)(q_\gamma t_m \tilde{\psi}_\beta) - (\tilde{q}_\alpha t^m \psi_\delta)(\tilde{q}_\gamma t_m \psi_\beta) \Big\} .
\end{aligned}$$

Here we dropped dots on indices as there is no confusion. Now we replace the two terms in the second line of the r.h.s. with similar terms with β, γ exchanged. This effect can be cancelled by doubling the two terms in the third line once the fundamental identities for $\epsilon^{\alpha\beta}$ are used, and we get the following

$$\begin{aligned}
\mathcal{L}_{\mathrm{Y}} &= i\pi\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} \Big\{ -(q_\alpha t^m \psi_\gamma)(q_\beta t_m \psi_\delta) - (\tilde{q}_\alpha t^m \tilde{\psi}_\gamma)(\tilde{q}_\beta t_m \tilde{\psi}_\delta) \\
&\qquad - (q_\alpha t^m \tilde{\psi}_\gamma)(q_\beta t_m \tilde{\psi}_\delta) - (\tilde{q}_\alpha t^m \psi_\gamma)(\tilde{q}_\beta t_m \psi_\delta) \\
&\qquad - 2(q_\alpha t^m \tilde{\psi}_\delta)(q_\gamma t_m \tilde{\psi}_\beta) - 2(\tilde{q}_\alpha t^m \psi_\delta)(\tilde{q}_\gamma t_m \psi_\beta) + 4(q_\alpha t^m \psi_\gamma)(\tilde{q}_\delta t_m \tilde{\psi}_\beta) \Big\} \\
&= -i\pi k_{mn} C^{\alpha\beta} C^{\gamma\delta} \left\{ \mathcal{J}^m_{\alpha\gamma} \mathcal{J}^n_{\beta\delta} - 2\mathcal{J}^m_{\alpha\gamma} \mathcal{J}^n_{\delta\beta} \right\} .
\end{aligned} \tag{A.1}$$

Potential. We begin by collecting some useful formulae. We start from

$$\begin{aligned} 0 &= (\mu^{mn})^{\alpha}{}_{\beta}(\mu_m)^{\beta}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} + (\mu^{mn})^{\beta}{}_{\beta}(\mu_m)^{\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} + (\mu^{mn})_{\gamma\beta}(\mu_m)^{\beta\alpha}(\mu_n)^{\gamma}{}_{\alpha} \\ &= 2(\mu^{mn})^{\alpha}{}_{\beta}(\mu_m)^{\beta}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} + (\mu^{mn})^{\beta}{}_{\beta}(\mu_m)^{\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} \\ &= f_{pmn}(\mu^p)^{\alpha}{}_{\beta}(\mu^m)^{\beta}{}_{\gamma}(\mu^n)^{\gamma}{}_{\alpha} + 2(\mu^{mn})^{\beta}{}_{\beta}(\mu_m)^{\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha}\,. \end{aligned} \tag{A.2}$$

Similar equalities hold if some q are replaced by $\tilde{q}$, which we express by putting dots to the indices. Putting dots to β we get

$$0 = 2(\mu^{mn})^{\alpha}{}_{\dot\beta}(\mu_m)^{\dot\beta}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} + (\mu^{mn})^{\dot\beta}{}_{\dot\beta}(\mu_m)^{\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha}\,. \tag{A.3}$$

Putting dots to α we get

$$\begin{aligned} 0 &= 2(\mu^{mn})^{\dot\alpha}{}_{\beta}(\mu_m)^{\beta}{}_{\gamma}(\mu_n)^{\gamma}{}_{\dot\alpha} + 2(\mu^{mn})^{\beta}{}_{\beta}(\mu_m)^{\dot\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\dot\alpha} + 2(\mu^{mn})^{\gamma}{}_{\beta}(\mu_m)^{\beta}{}_{\dot\alpha}(\mu_n)^{\dot\alpha}{}_{\gamma} \\ &= -(\mu^{mn})^{\dot\beta}{}_{\dot\beta}(\mu_m)^{\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\alpha} + 3(\mu^{mn})^{\beta}{}_{\beta}(\mu_m)^{\dot\alpha}{}_{\gamma}(\mu_n)^{\gamma}{}_{\dot\alpha} + f_{mnp}(\mu^m)^{\gamma}{}_{\beta}(\mu^n)^{\beta}{}_{\dot\alpha}(\mu^p)^{\dot\alpha}{}_{\gamma}\,. \end{aligned} \tag{A.4}$$

Here the first term was rewritten using $(\mu^{mn})^{\alpha}{}_{\dot\beta} = -(\mu^{nm})_{\dot\beta}{}^{\alpha}$ and (A.3), and the third term was decomposed into symmetric and antisymmetric parts in mn. Now consider

$$I \equiv 2f_{mnp}(\mathcal{M}^m)^{\alpha}{}_{\beta}(\mathcal{M}^n)^{\beta}{}_{\gamma}(\mathcal{M}^p)^{\gamma}{}_{\alpha} + 9(\mathcal{M}^{mn})^{\alpha}{}_{\alpha}(\mathcal{M}_m)^{\beta}{}_{\gamma}(\mathcal{M}_n)^{\gamma}{}_{\beta}\,. \tag{A.5}$$

Expanding this into μ's and using (A.4) and (A.2) we find

$$\begin{aligned} I = &-\frac{5}{2}f_{mnp}(\mu^m)^{\alpha}{}_{\beta}(\mu^n)^{\beta}{}_{\gamma}(\mu^p)^{\gamma}{}_{\alpha} - \frac{5}{2}f_{mnp}(\mu^m)^{\dot\alpha}{}_{\dot\beta}(\mu^n)^{\dot\beta}{}_{\dot\gamma}(\mu^p)^{\dot\gamma}{}_{\dot\alpha} \\ &+15(\mu^{mn})^{\dot\alpha}{}_{\dot\alpha}(\mu_m)^{\beta}{}_{\gamma}(\mu_n)^{\gamma}{}_{\beta} + 15(\mu^{mn})^{\alpha}{}_{\alpha}(\mu_m)^{\dot\beta}{}_{\dot\gamma}(\mu_n)^{\dot\gamma}{}_{\dot\beta}\,. \end{aligned} \tag{A.6}$$

Hence the potential term in (2.4) is $\mathcal{L}_{\text{pot}} = -V = \pi^2 I/15$.

Supersymmetry transformation. Let $\eta_{\alpha\dot\alpha}$ be the parameter of $\mathcal{N}=4$ supersymmetry. We define $\eta_{\dot\alpha\alpha} = -\eta_{\alpha\dot\alpha}$, and introduce the 4×4 matrix valued spinor

$$\hat{\eta}_{\alpha}{}^{\beta} = \begin{pmatrix} 0 & \eta_{\alpha}{}^{\dot\beta} \\ \eta_{\dot\alpha}{}^{\beta} & 0 \end{pmatrix}. \tag{A.7}$$

Rewriting the $\mathcal{N}=4$ transformation law (2.5) in terms of Sp(4) multiplets $\Phi^A_\alpha, \Psi^A_\alpha$ and $\hat{\eta}_{\alpha}{}^{\beta}$, we easily obtain (2.13).

Sp(4) ***R*-symmetry.** The R-symmetry acts on matter fields as

$$\Phi'^A_\alpha = U_\alpha{}^\beta \Phi^A_\beta\,, \qquad \Psi'^A_\alpha = U_\alpha{}^\beta \Psi^A_\beta\,. \tag{A.8}$$

U is unitary and satisfies $U^* = CUC^{-1}$, $U^T C U = C$. They are equivalently SO(5) spinors with charge conjugation matrix C. The $\mathcal{N}=4$ supersymmetry parameter $\hat\eta$ satisfies,

$$C\hat\eta C^{-1} = \hat\eta^T\,, \quad (\hat\eta^*) = C\hat\eta C^{-1}\,, \quad \mathrm{Tr}[\hat\eta] = 0\,, \quad \Gamma_5\hat\eta\Gamma_5 = -\hat\eta\,, \tag{A.9}$$

where $(\Gamma_5)_\alpha{}^\beta = \mathrm{diag}(+1,+1,-1,-1)$. The Sp(4) R-invariance removes the last condition and uplifts the supersymmetry to $\mathcal{N}=5$.

JHEP09(2008)002

A.2 The $\mathcal{N} = 6$ case

Yukawa term. Substituting (3.9) in the Yukawa terms of the $\mathcal{N} = 5$ Lagrangian (2.12) and expanding, we get the $\mathcal{N} = 6$ Yukawa term $\mathcal{L}_{\rm Y} = i\pi I_{\rm Y}$.

$$\begin{aligned} I_Y &= 2(J^n)_{\alpha\beta}(\bar{J}_n)^{\alpha\beta} - 4(J^n)_{\alpha\beta}(\bar{J}_n)^{\beta\alpha} \\ &\quad +(J^n)_{\alpha\beta}(J_n)_{\gamma\delta}(2C^{\alpha\delta}C^{\beta\gamma} - C^{\alpha\gamma}C^{\beta\delta}) + (\bar{J}^n)^{\alpha\beta}(\bar{J}_n)^{\gamma\delta}(2C_{\alpha\delta}C_{\beta\gamma} - C_{\alpha\gamma}C_{\beta\delta}). \end{aligned} \tag{A.10}$$

Using the fundamental identity (3.6) and the Sp(4) identity (2.32) we get the Yukawa terms in (3.12).

Potential. The potential term is $\mathcal{L}_{\rm pot} = -V = \pi^2 I/15$, where

$$I = 2f_{mnp}(\mathcal{M}^m)^\alpha{}_\beta(\mathcal{M}^n)^\beta{}_\gamma(\mathcal{M}^p)^\gamma{}_\alpha + 9(\mathcal{M}^{mn})^\gamma{}_\gamma(\mathcal{M}_m)^\alpha{}_\beta(\mathcal{M}_n)^\beta{}_\alpha \equiv 2I_1 + 9I_2\,. \tag{A.11}$$

Substituting (3.7) and (3.8) we obtain the intermediate results,

$$\begin{aligned} 2I_1 &= -4f_{mnp}(M^m)^\alpha{}_\beta(M^n)^\beta{}_\gamma(M^p)^\gamma{}_\alpha + 12f_{mnp}(M^m)^\alpha{}_\beta(M^n)^\beta{}_\gamma C^{\gamma\rho}(M^p)^\sigma{}_\rho C_{\sigma\alpha}, \\ 9I_2 &= -36(M^{mn})^\gamma{}_\gamma(M^m)^\alpha{}_\beta(M^n)^\beta{}_\alpha + 36(M^{mn})^\gamma{}_\gamma(M^m)^\alpha{}_\beta C^{\beta\rho}(M^n)^\sigma{}_\rho C_{\sigma\alpha}. \end{aligned} \tag{A.12}$$

In the right hand side of both equations, the first term is itself SU(4) invariant. The remaining terms, which we denote as $12X_1$ and $36X_2$, should combine into an SU(4) invariant. To see this, we introduce a new SU(4) invariant term and decompose it using the identities (2.32) and (3.6),

$$Z \equiv \epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\rho\sigma\tau}(M^{mn})^\beta{}_\rho(M_m)^\gamma{}_\sigma(M_n)^\delta{}_\tau = 4X_2 - X_3 + 2X_4 + 2X_5\,, \tag{A.13}$$

$$X_3 \equiv (M_m)^\alpha{}_\beta(M_n)^\beta{}_\gamma C^{\gamma\rho}(M^{mn})^\sigma{}_\rho C_{\sigma\alpha}\,, \tag{A.14}$$

$$X_4 \equiv (M_m)^\gamma{}_\gamma(M^{nm})^\alpha{}_\beta C^{\beta\rho}(M_n)^\sigma{}_\rho C_{\sigma\alpha}\,, \tag{A.15}$$

$$X_5 \equiv (M_n)^\gamma{}_\gamma(M^{nm})^\alpha{}_\beta C^{\beta\rho}(M_m)^\sigma{}_\rho C_{\sigma\alpha}\,. \tag{A.16}$$

Inserting $f^{mn}{}_p t^p = [t^m, t^n]$ into different t^m factors in X_1, one can show

$$X_1 = X_2 + X_3 = -X_2 + X_4 = -X_2 + X_5, \tag{A.17}$$

from which one can easily find $12X_1 + 36X_2 = 4Z$ as expected. Note also that

$$\begin{aligned} Z &= (M^{mn})^\beta{}_\rho(M_m)^\gamma{}_\sigma(M_n)^\delta{}_\tau\left\{\delta^\rho_\beta\delta^\sigma_\gamma\delta^\tau_\delta \pm (\text{5 other terms})\right\} \\ &= 4\mathrm{Tr}(M^{mn}M_nM_m) + 2\mathrm{Tr}(M^{mn}M_mM_n)\,, \end{aligned} \tag{A.18}$$

where the trace is with respect to the SU(4) indices and the fundamental identity was used. The potential term $\mathcal{L}_{\rm pot} = -V = \pi^2 I/15$ finally becomes

$$\begin{aligned} I &= -4f_{mnp}\mathrm{Tr}(M^mM^nM^p) + 44\mathrm{Tr}(M_{mn}M^mM^n) + 16\mathrm{Tr}(M^{mn}M_nM_m) \\ &= 40\mathrm{Tr}(M^{mn}M_mM_n) + 20\mathrm{Tr}(M^{mn}M_nM_m) \\ &= 60\mathrm{Tr}(M^{mn}M_mM_n) - 20f^{mnp}\mathrm{Tr}(M_mM_nM_p)\,, \\ \mathcal{L}_{\rm pot} &= 4\pi^2\mathrm{Tr}(M^{mn}M_mM_n) - \frac{4\pi^2}{3}f^{mnp}\mathrm{Tr}(M_mM_nM_p)\,. \end{aligned} \tag{A.19}$$

B. Mass deformation

In this section, we present a supersymmetry preserving mass deformation of the $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories. It was shown in [13] that the extended Gaiotto-Witten theories allow a mass deformation which preserves the whole $\mathcal{N} = 4$ supersymmetry and SO(4) R-symmetry.

The mass deformation adds the following terms to the Lagrangian [13],

$$\begin{aligned}\mathcal{L}_{\text{mass}} = & -\frac{\omega_{AB}}{2}\left(m^2\epsilon^{\alpha\beta}q^A_\alpha q^B_\beta + m^2\epsilon^{\dot\alpha\dot\beta}\tilde q^A_{\dot\alpha}\tilde q^B_{\dot\beta} + im\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}\psi^B_{\dot\beta} - im\epsilon^{\alpha\beta}\tilde\psi^A_\alpha\tilde\psi^B_\beta\right)\\ & -\frac{2\pi}{3}mk_{mn}\left\{(\mu^m)_{\alpha\beta}(\mu^n)^{\beta\alpha} - (\tilde\mu^m)_{\dot\alpha\dot\beta}(\tilde\mu^n)^{\dot\beta\dot\alpha}\right\},\end{aligned} \tag{B.1}$$

and the supersymmetry transformation rules,

$$\delta_{\text{mass}}\psi^A_{\dot\alpha} = mq^A_\alpha\eta^\alpha{}_{\dot\alpha}, \qquad \delta_{\text{mass}}\tilde\psi^A_\alpha = m\tilde q^A_{\dot\alpha}\eta_\alpha{}^{\dot\alpha}. \tag{B.2}$$

We will generalize this result to the $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories of this paper. We will find that supersymmetries are all preserved, but the R-symmetry gets partially broken.

$\mathcal{N} = 5$ mass deformation

Using the notations introduced in section 2.2, the $\mathcal{N} = 4$ mass deformed Lagrangian (B.1) is rewritten as

$$\begin{aligned}\mathcal{L}_{\text{mass}} = & -\frac{\omega_{AB}}{2}\left(m^2\Phi^A_\alpha C^{\alpha\beta}\Phi^B_\beta - im\Psi^A_\alpha(C\Gamma_5)^{\alpha\beta}\Psi^B_\beta\right)\\ & +\frac{2\pi m}{3}(\mathcal{M}_m)_\alpha{}^\beta(\mathcal{M}^m)_\beta{}^\gamma(\Gamma_5)_\gamma{}^\alpha,\end{aligned} \tag{B.3}$$

where the matrix Γ_5 is defined by $(\Gamma_5)_\alpha{}^\beta = \text{diag}(+1,+1,-1,-1)$. The mass deformation to the supersymmetry transformation rule is

$$\delta_{\text{m}}\Psi^A_\alpha = m(\Gamma_5)_\alpha{}^\gamma\eta_\gamma{}^\beta\Phi^A_\beta. \tag{B.4}$$

The explicit dependence on Γ_5 implies that the SO(5) R-symmetry is broken down to the SO(4) subgroup. Nevertheless, one can show that the mass deformed theory is invariant under the whole $\mathcal{N} = 5$ supersymmetry deformed by (B.4). This is consistent with previous results [18, 17] on mass deformation of the $\mathcal{N} = 8$ BLG model of SO(4) gauge group. For the BLG model, the mass terms did not break any supersymmetry despite the R-symmetry breaking.

Checking the supersymmetry. Let us sketch the proof of the full $\mathcal{N} = 5$ invariance. We work order by order in m. The $\mathcal{O}(m^2)$ terms in $\delta\mathcal{L}$ arise from δ of the boson mass term and δ_{mass} of the fermion mass term.

$$\begin{aligned}\delta\mathcal{L}|_{m^2} &= -\omega_{AB}m^2\Phi^A_\alpha C^{\alpha\beta}\delta\Phi^B_\beta + im\omega_{AB}\Psi^A_\alpha(C\Gamma_5)^{\alpha\beta}\delta_{\text{mass}}\Psi^B_\beta\\ &= -i\omega_{AB}m^2\Phi^A_\alpha C^{\alpha\beta}\eta_\beta{}^\gamma\Psi^B_\gamma + im^2\omega_{AB}\Psi^A_\alpha(C\Gamma_5)^{\alpha\beta}(\Gamma_5)_\beta{}^\gamma\eta_\gamma{}^\delta\Phi^B_\delta \;=\; 0.\end{aligned} \tag{B.5}$$

The $\mathcal{O}(m)$ terms fall into two types, one proportional to $m\eta\Psi D\Phi$ and the other to $m\eta\Psi\Phi^3$. The terms of the first type arise from δ of the fermionic mass term and δ_{mass} of the fermionic kinetic term, and are easily shown to cancel each other. The remaining $\mathcal{O}(m)$ terms are given by

$$\begin{aligned}\delta\mathcal{L}|_m = & \frac{2\pi im}{3}\Big((\Gamma_5)_\alpha{}^\beta\eta_\beta{}^\gamma(\mathcal{M}_m)_\gamma{}^\delta(\mathcal{J}^m)_\delta{}^\alpha - 2(\Gamma_5)_\alpha{}^\beta(\mathcal{M}_m)_\beta{}^\gamma\eta_\gamma{}^\delta(\mathcal{J}^m)_\delta{}^\alpha\Big)\\ & +\frac{2\pi im}{3}\Big(2(\Gamma_5)_\gamma{}^\alpha(\mathcal{M}_m)_\alpha{}^\beta\eta_\beta{}^\delta(\mathcal{J}^m)_\delta{}^\gamma + 2(\Gamma_5)_\gamma{}^\alpha(\mathcal{M}_m)_\alpha{}^\beta(\mathcal{J}^m)_\beta{}^\delta\eta_\delta{}^\gamma\Big)\\ & +\frac{2\pi im}{3}\Big(3(\Gamma_5)_\gamma{}^\epsilon\eta_\epsilon{}^\rho(\mathcal{M}_m)_\rho{}^\alpha(\mathcal{J}^m)_\alpha{}^\gamma - 6(\Gamma_5)_\epsilon{}^\gamma(\mathcal{J}^m)_\gamma{}^\alpha(\mathcal{M}_m)_\alpha{}^\rho\eta_\rho{}^\epsilon\Big)\,, \qquad \text{(B.6)}\end{aligned}$$

where the three lines in the right hand side correspond respectively to δ of the fermion mass term, δ of the quartic potential term and δ_{mass} of the Yukawa interaction. It appears difficult to show that this vanishes, but since we know it vanishes when $\eta_\alpha{}^\beta$ is proportional to $\Gamma_1,\ldots,\Gamma_4$, we only need to check that it vanishes when $\eta\sim\Gamma_5$.

$\mathcal{N}=6$ mass deformation

Following the lifting procedure in section 3, one can show that the mass deformed Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\text{mass}} = & -m^2\bar\Phi_A^\alpha\Phi_\alpha^A + im\bar\Psi_{\alpha A}(C\Gamma_5C)^\alpha{}_\beta\Psi^{A\beta}\\ & +\frac{2\pi m}{3}(M^m)^\alpha{}_\beta(M_m)^\gamma{}_\delta\Big\{-2\delta_\alpha{}^\beta(\Gamma_5)_\gamma{}^\delta + (\Gamma_5C)_{\alpha\gamma}C^{\beta\delta} + C_{\alpha\gamma}(C\Gamma_5)^{\beta\delta}\Big\}\,. \qquad \text{(B.7)}\end{aligned}$$

We regard Φ_α^A as a SO(6) left-handed spinor and $\Psi^{A\alpha}$ as a right-handed spinor. Introducing the six-dimensional Gamma matrices

$$\hat\Gamma_I = \begin{pmatrix} 0 & (\bar\rho_I)_{\alpha\beta} \\ (\rho_I)^{\alpha\beta} & 0\end{pmatrix}, \qquad \rho_I = (C\Gamma_I, iC), \qquad \bar\rho_I = (\Gamma_IC, -iC)\,, \qquad \text{(B.8)}$$

one can write $\mathcal{L}_{\text{mass}}$ in the following form

$$\begin{aligned}\mathcal{L}_{\text{m}} = & -m^2\bar\Phi_A^\alpha\Phi_\alpha^A - m\bar\Psi_{\alpha A}(\rho_{56})^\alpha{}_\beta\Psi^{A\beta}\\ & +\frac{2\pi im}{3}(M^m)^\alpha{}_\beta(M_m)^\gamma{}_\delta\Big\{2\delta_\alpha{}^\beta(\bar\rho_{56})_\gamma{}^\delta - (\bar\rho_5)_{\alpha\gamma}(\rho_6)^{\beta\delta} + (\bar\rho_6)_{\alpha\gamma}(\rho_5)^{\beta\delta}\Big\}\,. \qquad \text{(B.9)}\end{aligned}$$

The deformation to the supersymmetry transformation rule is

$$\delta_{\text{mass}}\Psi^{\alpha A} = m(C\Gamma_5C)^\alpha{}_\beta\eta^{\beta\gamma}\Phi_\gamma^A = im(\rho_{56})^\alpha{}_\beta\eta^{\beta\gamma}\Phi_\gamma^A\,. \qquad \text{(B.10)}$$

The mass deformation breaks the SO(6) R-symmetry down to SO(4) $\times$ SO(2). But this deformation preserves all $\mathcal{N}=6$ supersymmetry, since the SO(2) relates the sixth supersymmetry with the fifth one which has been shown to be the symmetry. See [44, 11] for related discussions.

References

[1] J.H. Schwarz, *Superconformal Chern-Simons theories*, *JHEP* **11** (2004) 078 [hep-th/0411077].

[2] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108].

[3] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955].

[4] J. Bagger and N. Lambert, *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[5] A. Gustavsson, *Algebraic structures on parallel M2-branes*, arXiv:0709.1260.

[6] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[7] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, $N = 8$ *superconformal Chern-Simons theories*, *JHEP* **05** (2008) 025 [arXiv:0803.3242].

[8] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple M2-branes*, *JHEP* **05** (2008) 105 [arXiv:0803.3803].

[9] N. Lambert and D. Tong, *Membranes on an orbifold*, arXiv:0804.1114.

[10] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256].

[11] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N = 6$ *superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, arXiv:0806.1218.

[12] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings and the theta-angle in* $N = 4$ *super Yang-Mills theory*, arXiv:0804.2907.

[13] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, $N = 4$ *superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [arXiv:0805.3662].

[14] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-matter theories*, *JHEP* **08** (2007) 056 [arXiv:0704.3740].

[15] C.-H. Ahn, K. Oh and R. Tatar, *Branes, orbifolds and the three dimensional* $N = 2$ *SCFT in the large-N limit*, *JHEP* **11** (1998) 024 [hep-th/9806041].

[16] A. Hanany, N. Mekareeya and A. Zaffaroni, *Partition functions for membrane theories*, arXiv:0806.4212.

[17] J. Gomis, A.J. Salim and F. Passerini, *Matrix theory of type IIB plane wave from membranes*, *JHEP* **08** (2008) 002 [arXiv:0804.2186].

[18] K. Hosomichi, K.-M. Lee and S. Lee, *Mass-deformed Bagger-Lambert theory and its BPS objects*, arXiv:0804.2519.

[19] Y. Honma, S. Iso, Y. Sumitomo and S. Zhang, *Scaling limit of* $N = 6$ *superconformal Chern-Simons theories and Lorentzian Bagger-Lambert theories*, arXiv:0806.3498.

[20] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general Lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012].

JHEP09(2008)002

[21] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, *$N = 8$ superconformal gauge theories and $M2$ branes*, arXiv:0805.1087.

[22] P.-M. Ho, Y. Imamura and Y. Matsuo, *$M2$ to $D2$ revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202].

[23] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Ghost-free superconformal action for multiple $M2$-branes*, *JHEP* **07** (2008) 117 [arXiv:0806.0054].

[24] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *Supersymmetric Yang-Mills theory from lorentzian three-algebras*, arXiv:0806.0738.

[25] B. Ezhuthachan, S. Mukhi and C. Papageorgakis, *$D2$ to $D2$*, *JHEP* **07** (2008) 041 [arXiv:0806.1639].

[26] S. Cecotti and A. Sen, *Coulomb branch of the lorentzian three algebra theory*, arXiv:0806.1990.

[27] M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons theories and AdS_4/CFT_3 correspondence*, arXiv:0806.1519.

[28] Y. Imamura and K. Kimura, *Coulomb branch of generalized ABJM models*, arXiv:0806.3727.

[29] J. Bhattacharya and S. Minwalla, *Superconformal indices for $\mathcal{N} = 6$ Chern-Simons theories*, arXiv:0806.3251.

[30] T. Nishioka and T. Takayanagi, *On type IIA Penrose limit and $N = 6$ Chern-Simons theories*, *JHEP* **08** (2008) 001 [arXiv:0806.3391].

[31] J.A. Minahan and K. Zarembo, *The Bethe ansatz for superconformal Chern-Simons*, arXiv:0806.3951.

[32] A. Armoni and A. Naqvi, *A non-supersymmetric large-N $3D$ CFT and its gravity dual*, arXiv:0806.4068.

[33] K.-M. Lee and H.-U. Yee, *New $AdS_4 \times X_7$ geometries with $N = 6$ in M-theory*, *JHEP* **03** (2007) 012 [hep-th/0605214].

[34] S. Lee, *Superconformal field theories from crystal lattices*, *Phys. Rev.* **D 75** (2007) 101901 [hep-th/0610204].

[35] S. Lee, S. Lee and J. Park, *Toric AdS_4/CFT_3 duals and M-theory crystals*, *JHEP* **05** (2007) 004 [hep-th/0702120].

[36] S. Kim, S. Lee, S. Lee and J. Park, *Abelian gauge theory on $M2$-brane and toric duality*, *Nucl. Phys.* **B 797** (2008) 340 [arXiv:0705.3540].

[37] C. Ahn, *Holographic supergravity dual to three dimensional $N = 2$ gauge theory*, arXiv:0806.1420.

[38] D. Gaiotto, S. Giombi and X. Yin, *Spin chains in $N = 6$ superconformal Chern-Simons-matter theory*, arXiv:0806.4589.

[39] S. Elitzur, A. Giveon, D. Kutasov and D. Tsabar, *Branes, orientifolds and chiral gauge theories*, *Nucl. Phys.* **B 524** (1998) 251 [hep-th/9801020].

[40] B. Feng and A. Hanany, *Mirror symmetry by O3-planes*, *JHEP* **11** (2000) 033 [hep-th/0004092].

JHEP09(2008)002

[41] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[42] S. Franco et al., *Dimers and orientifolds*, *JHEP* **09** (2007) 075 [arXiv:0707.0298].

[43] M. Berkooz et al., *Anomalies, dualities and topology of $D = 6$ $N = 1$ superstring vacua*, *Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184].

[44] I. Bena, *The M-theory dual of a* 3 *dimensional theory with reduced supersymmetry*, *Phys. Rev.* **D 62** (2000) 126006 [hep-th/0004142].